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LIMITING EQUILIBRIUM OF BRITTLE SOLIDS WITH FRACTURES

by

V. V. Panasyuk

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LIMITING EQUILIBRIUM OF BRITTLE SOLIDS WITH FRACTURES

By: V. V. Panasyuk

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

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* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH

DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}
rot	curl
lg	log

The following monograph set forth the bases of theory of quasi-stationary equilibrium of brittle solids with fractures, the development of works in this region is analyzed. Results of studies by the author on the theory of cracks propagation in a deformed brittle solid are generalized, and calculation model diagrams are formulated for solving such problems; the solutions to new plane and three-dimensional problems about maximum equilibrium of brittle solids with fractures are given; an attempt is made to formulate elements of a theory of brittle rupture of deformable solids with fracture defects.

This book is intended for scientists and engineer-technicians studying questions of the strength of solids; it can be useful also to teachers, graduate students and students of higher educational institutions specializing in mechanics and solid state physics.

Editor-in-Chief, Academician of the Academy of Sciences Ukrainian SSR G. Yu. Savin.

FROM THE EDITOR

The monograph offered to the attention of the reader is dedicated to consideration of elements of the theory of deformation and rupture of solid bodies with peaked fracture-type stress concentrators (rectilinear or curvilinear narrow slots – cracks, holes with angular on the contour, etc.). In the deformation of a solid weakened by such stress concentrators in the neighborhood of the concentrator point there appears a high intensity of stresses, which causes either the plastic flow of the material, or the propagation of a brittle crack. Study of these phenomena and determination of the limit, i.e., permissible from the point of view of the strength of the body, external loads (especially, when there is a possibility of a brittle crack) is of great scientific and practical interest. This is especially important for the brittle and quasi-brittle rupture of deformed solids.

Within the bounds of the mechanics of a continuous medium studies in this direction are a further development of the problem of stress concentration in a deformable elastic body with a special form of the stress concentrator – cracks.

The theory of fractures appeared more than 40 years ago. The first work was done by A. A. Griffith. However, fracture theory has been considerably developed only during the last two decades, in particular, in the work of the Soviet researchers S. A. Khristianovich,

G. I. Barenblatt, M. Ya. Leonov, G. P. Cherepanov and their followers. An important cycle of research in fracture theory is carried out also in the Ukraine by V. I. Mossakovskiy, V. V. Panasyuk and others. There are still no surveys on this area of mechanics. The first such attempt is the monograph of V. V. Panasyuk. The author is not limited to an account of new data on fracture theory obtained in his and his colleagues' work, but, synthesizing the results of other researchers, in a single plan sets forth the basic achievements of the theory of quasi-stationary equilibrium of deformable brittle bodies with crack-type stress concentrators.

The great urgency of research in fracture theory for the problem of brittle rupture of solids, and also the scientific novelty of the problem assure us that the monograph of V. V. Panasyuk will be of great benefit to persons working in the applied theory of elasticity and theory of strength calculation of elements in engineering construction and will promote further development of research in fracture theory. It will be useful to teachers, graduate students and students of higher educational institutions specializing in the mechanics of deformable solids.

Academician of the Academy of Sciences Ukrainian SSR G. Yu. Savin

PREFACE

The strength of real solids is determined not only by their physicochemical nature, but also significantly depends on imperfections in their structure. The structure of real solids always has different type of defects – stress concentrators, such as, for example, micro- and macrocracks, different origin of a cavity and inclusion, grain boundaries and blocks of structure, accumulation of dislocations, vacancies and others.

In the process of deformation of a solid in the environment of such defects occurs a high concentration of stresses which leads to the formation of nuclei and growth of fractures already in the solid, i.e., to local or complete destruction of the solid. In a deformable solid such defects lead to local or complete destruction as a result of the formation and propagation of a main (the most dangerous) rupture crack. As experiment has shown, such a phenomenon is especially characteristic for brittle (linear-elastic) or almost brittle (quasi-brittle) destruction of deformable solids.

Thus, development of theory and methods of determination of the resistivity of a material to the development of fractures in it, and also the size of the limit (destroying) load for a deformable solid with crack-type imperfections (peaked cavities – slots) is one of the important stages in building a general theory of

deformation and destruction of solids, in particular a theory of brittle or quasi-brittle rupture. The creation of such a theory has great scientific and applied value, therefore the efforts of many researchers are directed towards the study of processes of the formation and development (propagation) of different imperfections – cracks in a deformable solid.

During the period of the last two decades research in the mechanics of deformable brittle bodies with stress concentrators appearing as peaked cavities – cracks has greatly developed. Calculation models are proposed for the solution of such problems, values of limit loads for certain cases of deformation of solid with shown defects are determined, prospects for the solution of new and similar problems are outlined.

This monograph attempts to illustrate in a single plan certain basic results of the statics of fragile bodies with fractures. The physical bases of a fracture theory of cracks are set forth; elements are developed of a theory of calculation of limit loads for a deformable brittle or quasi-brittle body, weakened by imperfections appearing as peaked cavities – cracks of a given configuration. Considerable attention is given to consideration of calculation models and setting up problems of fracture theory as problems of the mechanics of a deformable solid; solution is given to certain new (plane and three-dimensional) problems of statics of a brittle body with crack-type imperfections and also certain elements of a theory of indestructability (strength) of a brittle body with fractures under plane stress are formulated.

The monograph sets forth basically results obtained by the author and his colleagues at the Physicomechanical Institute of the Academy of Sciences Ukrainian SSR. The author notes that important values for development of his research in fracture theory resulted from the collaboration with Academician of the Academy of Sciences Kirgiz Soviet Socialist Republic M. Ya. Lenov, under whose guidance the first work of author in this region was

carried out, and he is thankful to him for the useful discussion of many questions set forth in the monograph.

The author expresses deep gratitude to Academician of the Academy of Sciences Ukrainian SSR G. N. Savin for the scientific editing of this monograph, and also for constant interest and attention to his work on fracture theory. The author is grateful to the colleagues of the department of Physical Bases of Strength of the Physicomechanical Institute of the Academy of Sciences Ukrainian SSR Cands. of Phys. and Math. Sci. L. T. Berezhnitskiy, P. M. Vitvitskiy, Eng. S. Ye. Kovchik for help in preparation of the book.

The author asks that all remarks and inquiries should be sent to: Kiev, Repina, 3, Publishing House "Naukova dumka."

INTRODUCTION

1. The Simplest Problem of Fracture Theory

The most important fracture of problems about the propagation of fractures in a deformable brittle body[1] is that in the neighborhood of the end region of the propagating fracture the material of the body always is deformed beyond the limits of elasticity. This makes impossible a quantitative description of deformation in such a region within the bounds of the classical positions of the theory of elasticity. A solution of problems of the theory of propagation of cracks in a deformable solid required the introduction of certain new ideas in the mechanics of solid elastic media and the creation of physically based calculation models.

Let us examine the simplest problem in the theory of propagation of cracks. Let us assume that a brittle unbounded plate, weakened by a through isolated linear crack of length $2l$ (Fig. 1), is extended by monotonically increasing external stresses p, applied in points at infinity of the plate and directed perpendicular to the plane of the crack. The thickness of the plate is taken as unity. It is necessary to determine the least stress $p = p_*$ (subsequently this stress will be examined as the limit strength, and the crack will be examined as a cut in an elastic continuum), after which a crack starts to spread and the plate loses its carrying capacity (ruptures).[2]

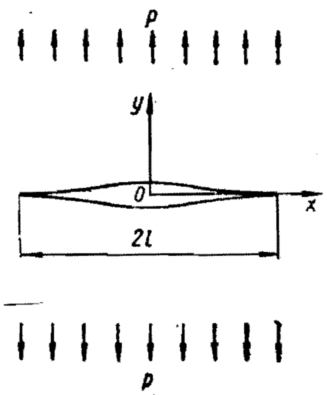


Fig. 1.

We introduce rectangular system of cartesian coordinates xOy (see Fig. 1) and try to solve the problem within the bounds of the classical theory of elasticity. Using certain procedures of the theory of elasticity [103], we find the formula for determination of stresses in the examined plate. Thus, in the plane of the crack (y = 0) the elastic rupturing stresses

$$\sigma_y(x, 0) = p\frac{|x|}{\sqrt{x^2 - l^2}} \quad (|x| \geqslant l). \tag{1}$$

Analysis of formula (1) shows that stresses $\sigma_y(x, 0)$ at $x = l$, i.e., rupturing elastic stresses in the dead-end part of the crack, are infinitely large for any length of the crack $l \neq 0$ and $p > 0$.

Inasmuch as a real body can sustain only finite tensile stresses, from this solution we get the following fact: for any small (but not equal to zero) stretching load[3] p a plate weakened by a crack should rupture immediately. However, data from experiments contradict this: the breaking load is a function of the dimensions crack in the body, and for a load smaller than a certain (maximum) value, the crack does not spread and body preserves its supporting power. Thus, formal application of the classical theory of elasticity to our problem leads to a physically unsupported result.

The above contradiction between the classical theory of elasticity and experimental data is explained by the following.

The model of the theory of elasticity incompletely considers the interpartial cohesive forces acting between the opposite sides of an opening crack at its dead-end part, i.e., in that region of the solid in which deformations exceeding limit of elastic deformations appear. Calculation of the forces of interpartial cohesion in the dead-end part of a crack considering equilibrium of the deformable solids weakened by the cracks is the main distinction of problems of the theory of cracks from classical problems of the theory of elasticity. The calculation of these forces permits constructing the necessary calculation model of a deformable solid, within the bounds of which can be found a solution to the problem of propagation of cracks which agrees with experiment.

2. Basic Positions of the Theory of Griffith

The problem about the rupture of a plate with a linear crack, when the plate is stretched by external stresses p, was studied by Griffith [190, 191]. To determine the maximum (rupture) load $p = p_*$ Griffith proposed the so-called power method. To clarify the essence of this method we will examine the potential energy of a deformed brittle solid weakened by a crack of length $2l$ and subjected to extension by external stresses p. This energy can be presented in the following form:

$$\Pi = \Pi_0 - W(p, l) + U(l),$$

where Π_0 – potential energy of deformable solid without crack; $W(p, l)$ – energy of elastic deformation caused by the opening of a crack of length $2l$ under the effect of stresses p; $U(l)$ – surface energy of crack.

Using this expression the power method of Griffith can be formulated as: for the propagation of a crack, i.e., to increase its length $2l$ at an assigned load $p = p_*$, it is necessary that with a growth of l energy Π of the deformed body does not increase.

Consequently, maximum strengths $p = p_*$ for a body with a crack of length $2l$ must satisfy the equation

$$\frac{\partial}{\partial l}[U(l) - W(l, p_*)] = 0. \quad (2)$$

Equation (2) is the fundamental equation of the Griffith theory of the propagation of cracks. On the basis of this equation Griffith investigated the breaking load for a plate with a linear crack (Fig. 1). The change of elastic energy $W(l, p)$ is calculated on the basis of certain relationships of the theory of elasticity (see for example, [103]) and is expressed by the following formulas[4]:

for plane deformation

$$W(l, p) = \frac{\pi p^2 l^2}{E}(1 - \nu^2); \quad (3)$$

for the plane generalized stressed state

$$W(l, p) = \frac{\pi p^2 l^2}{E},$$

where E — Young's modulus; ν — Poisson's ratio.

The surface energy of the crack in this case is determined by the equality

$$U(l) = 4l\gamma, \quad (4)$$

where γ — density of effective surface energy of material.

Substituting equality (3) and (4) into equation (2), after simple transformations we obtain the formula for determination of limit strengths $p = p_*$:

for plane deformation

$$p_* = \sqrt{\frac{2E\gamma}{\pi(1-\nu^2)l}}; \qquad (5)$$

for the plane generalized stressed state

$$p_* = \sqrt{\frac{2E\gamma}{\pi l}}. \qquad (5a)$$

Note. If at a given crack length $2l$ external stress reaches $p = p_*$, its further increase (no matter how small) will lead to propagation of the crack, i.e., to an increase of its length. Moreover the crack propagation will be spontaneous (unstable), since for all linear cracks of length $2l_1 > 2l$ limit strength $p_{*1} < p_*$. Thus, for the examined problem (see Fig. 1) after the external stress reaches $p = p_*$ crack propagation becomes unstable, consequently, this is the rupture stress.

Equations (5) and (5a) are the Griffith formulas. They can also be written as:

$$p_* \sqrt{l} = \text{const.} \qquad (5b)$$

Equality (5b) was checked by Griffith experimentally in the following way. In the walls of glass spherical retorts and cylindrical tubes, whose diameter was sufficiently great, and whose wall thickness was respectively 0.25 and 0.5 mm, cracks of different length formed. Then the retort and the tube were annealed at 450°C to remove residual stresses. The retort and tube were loaded with internal hydraulic pressure up to rupture and the rupture stress p_* was measured for every crack of length $2l$. Results (Table 1) confirmed the constancy of $p_* \sqrt{l}$ according to formula (5b) and thereby the theoretical diagram of Griffith [190].

Table 1.

Spherical retort				Tube			
Length of crack $2l$, cm	Diameter cm	Rupture stress		Length of crack $2l$, cm	Diameter cm	Rupture stress	
		p_*, kg/cm²	$p_* \sqrt{l}$			p_*, kg/cm²	$p_* \sqrt{l}$
0,371	3,78	594	258	0,635	1,50	465	262
0,685	3,88	428	250	0,815	1,81	406	253
1,370	4,06	331	274	0,965	1,88	361	250
2,260	5,07	251	266	0,710	1,55	450	267
				0,660	1,57	463	265
				0,762	1,55	423	259

In the theoretical diagram of Griffith during the derivation of equation (2) the action of interpartial cohesive forces in the dead-end part of a crack is determined integrally (by considering the balance of surface and elastic energy of a crack during its propagation).

Such an approach to the problem of propagation of cracks in a deformable brittle body, although it makes it possible to determine the limit stress, does not permit solving other questions of fracture theory: stresses in the dead-end part of a crack, structure of the edge of this part of a crack, etc. During calculation by formulas (5) and (5a) the following physically unjustified peculiarity also appears with a decrease of crack dimensions to zero the limit stress approaches infinity, although in this case (at $l = 0$) the rupture load of a faultless material should be finite. This phenomenon cannot be properly explained within the bounds of the theoretical diagram of Griffith.

Below the propagation of cracks will be formulated as a "power" problem of mechanics (without drawing in the power method of Griffith) and on this basis will be given a more than general solution of the simplest problems of fracture theory — solutions free from these deficiencies.

3. Short Survey of the Development of Research in Fracture Theory

Studies of the elastic equilibrium of an unbounded isotropic plane weakened by an elliptical cut (in particular, by linear cut), were first carried out by Koloson [60] (1909), and then Inglis [194] (1913) and Muskhelishvili [103] (1919). The most general and effective solution of the problem is given in a work of Muskhelishvili, which was of great value to subsequent studies of the propagation of cracks. However, these works still do not examine questions of the actual theory of the propagation of cracks in a deformable solid.

The first work on a theory of the propagation of cracks in a deformable solid was done by Griffith [190] (1920), [191] (1924), in which he developed the idea of the necessity of calculating forces of interpartial cohesion in the dead-end part of the crack. This permitted Griffith to formulate the so-called power method of determining the limit equilibrium state of a brittle body (plate) with a crack and for the first time to find a solution to the problem of the breaking load for a solid with an isolated linear crack (see Fig. 1).

Griffith also was the first to attempt an analysis of the structure of the edge of a crack in its dead-end region. This analysis, however, was on the basis of a classical solution of the theory of elasticity (neglecting forces of weakened interpartial connections in the dead-end part of the crack) on the assumption that the crack is elliptical. Such an analysis of the structure of the dead-end part of a crack, as will be shown below, cannot be recognized as correct.

The structure of the edge of a developed crack until recently did not have a proper mathematical analysis. But, nevertheless, certain researchers, proceeding from physical prerequisites, noted that cracks in a deformable solid are not ellipsoid (even with a very small radius of curvature on the ends). In the dead-end part

the sides of a developed crack must be closed smoothly, since the distance between the sides of a crack should gradually pass into an interatomic distance. Such ideas about the structure of real cracks (wedge-shaped slots) were first advanced in the work of P. A. Rebinder [147] in connection with analysis of the influence of physicochemical processes of a surface active medium on the deformation and rupture of solids.

Soon after the publishing of the work of Griffith Smekal published an article [220] discussing the Griffith theory and certain aspects of its application. In particular, Smekal attempted to explain on the basis of this theory the causes of such a large (two to three orders) deviation between the observed (technical) strength of macroscopic solids and their theoretical strength, emanating from the theory of the interatomic bond in a crystal lattice. The cause of this deviation, as it is known, is considered to be fracture-type imperfections in the real solid.

In a 1923 work of Wolf [228] certain calculations of Griffith were definitized [190], and also his method for solving the pure bend of a strip (beam) with an internal crack is used[5] (Fig. 2). Furthermore, this work examines the interconnection between the Griffith theory of rupture and previous phenomenological theories of strength.

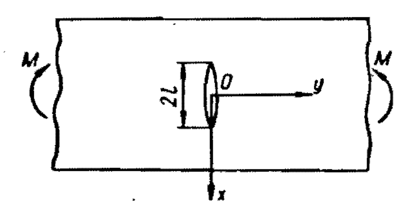


Fig. 2.

The same period includes the work of A. F. Ioffe [52] and Ya. I. Frenkel' [162], which contain an analysis of certain questions of the theory of the strength of solids and also discuss results of the studies by Griffith. Experiments of A. F. Ioffe and his

colleagues (see [52] on the strength of crystals showed that crystals of rock salt (NaCl) which break up in water after their surface (most defective) layers dissolve possess a high tear resistance. In separate cases this strength reached $160 \cdot 9.81$ N/mm^2, i.e., approached theoretical values ($200 \cdot 9.81$ N/mm^2). These experiments were the first to show the reality of calculation of theoretical strength with respect to forces of interpartial cohesion.

A work of I V. Obreimov [211] (1930) examines crack propagation when a thin platelet 1 mica is split from investigated sample 2 with the help of sliding wedge 3 (Fig. 3). On the basis of methods of approximation of the theory of the bending of thin beams this work sets up a formula connecting the density of the effective surface energy of a material, its elastic constants and the parameters of the developed crack. Calculation and experimental data, obtained in [211], subsequently were definitized in works [45, 62].

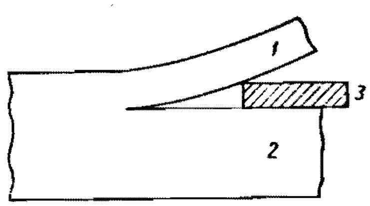


Fig. 3.

Of the subsequent research infracture theory it is necessary to note the work of Westergaard [223, 224]. These works present a method of solving classical problems of the theory of elasticity concerning the extension and compression of plates with cracks, analyze the distribution of stresses in a plate with semi-infinite crack and note the possibility of a limit stress in the neighborhood of the dead-end part of a crack. However, in the works of Westergaard the conclusion concerning the finiteness of stresses in the neighborhood of the dead-end part of a crack is not connected with the determination of limit load for a plate weakened by cracks.

In 1945 Mott [208] examined the Griffith problem as a dynamic problem of fracture theory and determined the rate of propagation of a crack. With this in mind he introduced into the power equation of Griffith (2) the kinetic energy of a unit surface of a crack:

$$\frac{\partial}{\partial l}\left\{k\rho\,(\dot{l})^2 l^2\left(\frac{p_*}{E}\right)^2-\frac{\pi p_*^2 l^2}{E}+4\gamma l\right\}=0, \qquad (6)$$

where ρ — density of material; k — dimensionless factor.

According to this equation the following formula is obtained for the determination of the rate ($dl/dt = \dot{l}$) of propagation of a macrocrack:

$$\dot{l}=\sqrt{\frac{\pi E}{k\rho}}\left(1-\frac{l_0}{l}\right)^{1/2}, \qquad (7)$$

where l_0 — initial half-length of crack; l — half-length of propagating crack.

Analysis of formula (7) shows that the speed of brittle rupture (propagation) of a crack increases with an increase of the length of the crack, composing in the limit ($l \to \infty$) a certain part of the propagation velocity of longitudinal waves.[6]

In 1946 Sack [216] used the power method of Griffith to solve the three-dimensional problem of the rupture of a brittle body weakened by a macroscopic disk-like crack, when the body is subjected at infinity to extension by a field of uniform stresses p, directed perpendicular to the plane of the crack.

Calculating the elastic energy W(a, p) of a crack[7] during the action of stresses p and using the power principle of Griffith,

Sack established the following formula for determining limit strength [216]:

$$p_* = \sqrt{\frac{\pi E \gamma}{2(1-\nu^2)a}}, \tag{8}$$

where a is the radius of the crack.

The Griffith and Sack were examined in 1947 by Elliot [184] without using the power principle. In this work along with calculation of the limit values $p = p_*$ an attempt is made to determine the structure of the edge of a crack. The classical problem of the theory of elasticity was examined for a plane with a linear cut of length $2l$ (see Fig. 1) or for an elastic space with a disk-like circular crack having radius a, when each of the bodies at infinity is stretched by stress p perpendicular to the plane of the cracks. For every problem normal tensile stress σ_y in planes parallel to the plane of the crack and a distance $\pm\frac{1}{2}\lambda_0$ from this plane were calculated, where λ_0 is the normal interatomic distance. Furthermore, normal shifts 2v appearing between these planes were calculated, and dependence $\sigma_y(2v)$ was constructed, including in the form of parameters stress p and dimensions of crack 2a (or $2l$).

A real brittle body weakened by a crack was examined as a body consisting of two semi-infinite blocks, a normal interatomic distance λ_0 from each other and attracted to one another by the forces of interpartial cohesion, acting only outside the region of the crack. For such a body the dependence $\sigma_y(2v)$, established as a result of the solution to the corresponding elastic problem, is examined as the true dependence of the change of forces of interpartial cohesion on the distance between surfaces of the shown blocks. It is assumed that the propagation of a crack (rupture of body) begins when the maximum value of stress $\sigma_y(2v)$ reaches the theoretical strength for a given material.

The values of maximum (breaking) load calculated in such a way in work [184] for the problems of Griffith and Sack turned out to be somewhat different (numerical coefficient) from the limit loads set for these problems in works of Griffith [191] and Sack [216]. The form of the developed crack, according to [184], is identified with the surface described by the shifts function $v(x, \pm\frac{1}{2}\lambda_0)$.

In his solution to problems on the propagation of cracks in a deformable solid Elliot wrongly disregarded the forces of interpartial attraction between the edges of a developed crack all over its surface. Inasmuch as the distance between the edges of the dead-end part of an opening crack is commensurable with interatomic distance, then in this part of the crack the forces of interpartial cohesion are still effective, which naturally must be considered in examining the maximum equilibrium state of a body. Thus, the distribution of stresses and shifts in the neighborhood of the dead-end part of a crack can essentially differ from the corresponding distributions obtained by Elliot. This makes it impossible to consider the strict analysis of the structure of the edge of an opening crack, given in work [184], and also selection of distance $\pm\frac{1}{2}\lambda_0$.

An investigation of the structure of the edge of a developed crack and a consideration of the theory Griffith occupied Ya. I. Frenkel' (1952) [163]. He examined a semi-infinite crack when a plate was peeled off and applied the theory of the bending of a rod to determination of the form of a crack. The approximate treatment of the problem and the assumed inaccuracies in calculations made it impossible to find a correct solution.[8]

A solution of new problems about crack propagation in a deformable brittle body (a plate) subjected at infinity to extension

by a uniform field of stresses was given by Willmore (1949) [227] and Bowie (1956) [177]. In work [227] within the bounds of the theoretical scheme of Griffith the problem of rupture of an unbounded plate, weakened by two collinear cracks is examined,[9] and in work [177] the problem of the breaking load for a plate with a circular hole, when on the contour of the hole one or two equal cracks emerge, directed perpendicular to the line of extension. The Bowie problem later was extended to the case of other holes in works A. A. Kaminskiy [53, 54]; an analogous problem about the quasi-brittle rupture of a plate with a circular (incipient) cracks was examined in works of P. M. Vitvitskiy and M. Ya. Leonov [30, 31].

An important stage in development of research in fracture theory was a work by Irwin [195-203] and Orowan [213, 214, 186], in which the theory of Griffith is extended to the case of brittle rupture of plastic materials. As it is known, direct application of the theory of Griffith in the study of brittle rupture of plastic materials is complicated because such rupture always is accompanied by microplastic deformations in the region of rupture of the body, although on the whole macroscopic rupture of the body remains brittle. Irwin and Orowan noted that during crack propagation in a quasi-brittle material microplastic deformations always are concentrated in the thin layer of material adjacent to the surface of the rupture crack. In connection with this it was proposed to change the theory of Griffith to study crack propagation in a quasi-brittle material thus: along with the surface energy of a material the specific work (energy of plastic deformation expended on formation of a unit of new crack surface is considered also. In this case the formula of Griffith for determination of the maximum stress takes the following form [214]:

$$p_* = \sqrt{\frac{2E(\gamma_0 + \gamma_n)}{\pi l}}, \tag{9}$$

where γ_{Π} — specific work of plastic deformation on a unit surface of a crack during quasi-brittle rupture; γ_0 — surface energy of material.

On the basis of X-ray structural investigations of the surface of brittle fractures of low-carbon steel it has been established that $\gamma_{\Pi} = 2\cdot10^6$ erg/cm^2. The surface energy of metals, as it is known [164], in order of value is 10^3 erg/cm^2, i.e., it is three orders less than γ_{Π}. Considering this, in formula (9), according to Orowan, it is possible to disregard γ_0. Then

$$p_* = \sqrt{\frac{2E\gamma_{\Pi}}{\pi l}}. \qquad (10)$$

The work of Fehlbeck and Orowan [186] describes special experiments by the study of brittle rupture of low-carbon steel. The essence of these experiments is that samples were made in the form of low-carbon steel plates weakened by a crack perpendicular to the lateral face. The cracks were made in the following way. In a plate a thin notch was made perpendicular to the lateral face, then the plate was cooled by liquid nitrogen and a wedge was pushed into the notch. As a result on the bottom of the notch a crack was formed. After that the upper part of the plate together with the surface notch was cut, and thus a plate weakened by a crack was obtained. To make a plate with an internal fracture two plates with surface fractures were welded so that the fractures met each other and formed a plate with an internal crack. Plates with surface and internal cracks were subjected to uniaxial extension up to rupture. For every length of the crack the external rupture stresses were determined.

Experiments showed [186] that the rupture stress is indeed inversely proportional to the square root of the initial crack length. However, values of γ_{Π} calculated according to experimental data and formula (10) (at $E = 2\cdot10^6\cdot9.81$ N/cm^2) are these: $3.4\cdot10^6$ for internal cracks and $4.9\cdot10^6$ erg/cm^2 for surface cracks. Obtained

values of γ_n exceed the value of $2 \cdot 10^6$ erg/cm^2, determined on the basis of X-ray structural measurements. Analysis of the surfaces of rupture cracks in this case showed that at room temperature the propagation of an initial crack when a plate is loaded follows this pattern: at first in its dead-end part there are considerable microplastic shifts (viscous break), and then a brittle (crystal) break occurs; after a certain distance the crystal break again changes to viscous.

Fehlbeck and Orowan [186] noted also that for a brittle rupture to develop in a plastic material an important role is played by the rate of propagation of the crack. While this rate is low, the brittle fracture of the plastic material cannot occur. When the rate of propagation of the crack reaches the necessary level, a brittle rupture occurs, which can change to viscous if the liberated elastic energy during the propagation of the crack is not sufficient for the continuous advance of the crack at the required rate.

The idea of Irwin and Orowan about quasi-brittle rupture considerably expanded the field of application of results of the theory of ideally brittle crack propagation.

In recent years the problem of appearance and development of the initial stages of plastic deformations during the propagation of a quasi-brittle crack has been examined on the basis of a new model of a solid in works of M. Ya. Leonov and his colleagues [28, 31, 74, 78, 149]. Experimental research in the propagation of cracks in metals is illustrated in a monograph of B. A. Drozdovskiy and Ya. B. Fridman [47], and also B. S. Kasatkin [57].

A general analysis of the power approach to study of brittle and quasi-brittle propagation of cracks in a deformable solid was made in 1958-1960 by Bueckner [179]. In [179] within the bounds of the theoretical scheme of Griffith-Irwin-Orowan it is noted

that the presence of mass (volume) forces does not complicate calculations during the determination of values of limit loads for a body with cracks, and also certain examples of the rupture of revolving disks with cracks are considered.

The solution to new problems about the propagation of cracks in a deformable brittle solid, and in particular, the investigation of cases when the field of external stresses is nonuniform for a long time was difficult due to the complexity of calculating the rate $dW(l, p)/dl$ at which elastic energy is released as a result of the opening of cracks. A certain development of investigations in this area is given in a work of Irwin [196-199, 203]. Formulas are presented which make it possible to connect the value of escape velocity of the elastic energy of a crack with the coefficients of the intensity[10] of elastic stresses near its dead-end part, and on this basis the problems are solved for new cases of the loading of a body and the location of cracks. In particular, work of Irwin [195, 197] affirms that the propagation of a crack in a brittle or quasi-brittle solid sets in when coefficient of intensity of the elastic stresses in the neighborhood of the contour of the crack reaches a certain constant level for a given material under assigned conditions, and works [179, 199, 225] examine the extension and bending of plates (strips) with a surface crack.

Essential progress in development of a theory of the equilibrium of solids weakened by cracks has been reached only in recent years in the works of Yu. P. Zheltov and S. A. Khristianovich [49], G. I. Barenblatt [4-9], M. Ya. Leonov [66, 67] and other Soviet researchers [10-15, 19-22, 28-32, 38, 48, 53-56, 58, 70-73, 75-77, 83-88, 91-94, 97-102, 110-139, 142, 152, 153, 166-169].

From 1958 to 1964 the authors of works [5-15, 31, 70-73, 110-118] advanced and illustrated by a series of examples the idea that for investigation of the propagation of cracks in a deformable brittle body, and also to construct a qualitative theory of these processes methods of the mathematical theory of elasticity and

model of a continuous elastic medium can be used if the interpartial cohesive forces acting in the dead-end part of the crack are examined as certain external forces. In this case the problem of the propagation of cracks can be formulated as a power problem of the mechanics of a deformable solid (without drawing in power considerations). With this approach to the propagation of cracks it is possible to show in particular that the rupture stresses in the dead-end part of a crack are finite, and the opposite edges smoothly close, i.e., to solve the structure of the edge of a crack, and also to obtain simpler equations of the maximum equilibrium of a deformable body with cracks.

It is necessary to note that affirmation of the finiteness of stresses in the dead-end part of a crack was first advanced in a work of Yu. P. Zheltov and S. A. Khristianovich [49] on the theory of the hydraulic rupture of a petroleum layer. Use of this hypothesis in works [4, 49] can solve certain problems about the dimensions of an opening crack in mountain massifs, when in crack a preassigned hydrostatic pressure is created (however, in the mentioned works interpartial cohesive forces are considered directly). The finiteness of stresses in the dead-end part of the crack in this case (conditions of a mountain massif) is ensured due to the action of compressive, forces of mountain pressure. Works [4, 49] had an essential value for understanding the structure of the edge of a crack and the development of a theory of equilibrium cracks.

During the equilibrium analysis of brittle bodies weakened by cracks, and also during the study of the brittle rupture of elements of engineering constructions (when volume forces of compression are neglibly small) it is very significant to consider forces of interpartial cohesion in the dead-end part of the crack. Works of G. I. Barenblatt (1959-61) [5-7] definitized and augmented the statement of problems on equilibrium macrocracks of brittle rupture and proposed a new theoretical system for solving such problems as problems in the machanics of a deformable solid. This

considerably promoted the development of research in fracture theory during the last few years. Basic positions of the theoretical system of G. I. Barenblatt are applied in the following section.

From 1959 to 1961 the studies of M. Ya. Leonov and the author [70, 71, 112] were published on formulation of certain calculated models of a real solid for the solution of problems about the propagation of cracks in a deformable brittle body.

The calculated model (subsequently called the δ_k-model of a brittle body), advanced in works [70, 71, 112], differs from the theoretical system of G. I. Barenblatt. On the basis of the applied model it is possible to investigate in a single plan the equilibrium state of a solid weakened by both micro- and macrocracks, and also to obtain data about the influence of microscopic cracks on the strength properties of a solid. In the case of macroscopic cracks the δ_k-model of a brittle solid in particular leads to the same results as the system of G. I. Barenblatt. A description and application of this model to certain problems of the theory of propagation of cracks in a deformable solid is given in this monograph.

Other approaches to problems about maximum equilibrium of brittle solids with cracks are expounded in works of M. Ya. Leonov and K. N. Rusinko [76, 77], Ya. B. Fridman and Ye. M. Morozov [165], G. P. Cherepanov [168, 169], L. M. Kachanov [58]. A rather extensive bibliography of works on fracture theory carried out by foreign researchers is contained in [175].

4. Continuation of Survey. Elements of the Theory of Macrocracks

To solve problems about the maximum equilibrium of brittle solids with macrocracks Soviet researchers have advanced new calculation systems based on the condition of finiteness of the stresses in the tip of a crack and certain hypotheses about the

forces of interaction between the opposite edges of a crack in its dead-end part. The most systematic presentation of such an approach for macrocracks is given in works of G. I. Barenblatt [8].

The theoretical system proposed by G. I. Barenblatt is based on the following hypotheses.

First hypothesis. Width d (Fig. 4) of the end region of a crack (i.e., region in which considerable forces of interpartial cohesion act) is small as compared to the dimensions of the entire crack.

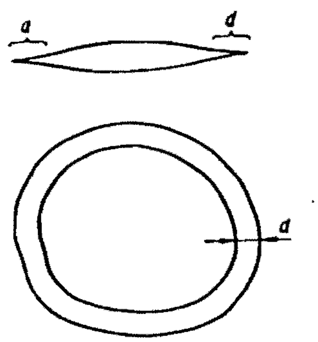


Fig. 4.

This hypothesis essentially determines the class of examined cracks, since this hypothesis to certain microcracks is inapplicable. Subsequently cracks for which this hypothesis is valid are called macrocracks.

Second hypothesis. The form of the normal section of the surface of a crack in the end region (and consequently, local distribution of cohesive forces over the surface of the crack) does not depend on the loads, and for a given material under these circumstances (temperature, composition and pressure of surrounding atmosphere, etc.) always is identical (by normal section here we understand the section of the plane normal with respect to the contour of the crack).

According to this hypothesis, the dead-end part of a crack during propagation should move forward, keeping its initial form. This is the so-called hypothesis of the autonomy of the end range of a crack [5]. It is applicable only for those points of the contour of a crack for which the highest possible intensity of cohesive forces is attained in the given conditions.

If on the contour of an equilibrium crack is there at least one point in which the maximum possible intensity of cohesive forces reached, the crack is called transient-equilibrium. The external stresses after the achievement of which an equilibrium crack passes into a transient-equilibrium crack are called maximum or critical.

Thus, the second hypothesis and the conclusions relying on it are applicable only for the investigation of the transient-equilibrium state of a crack in a deformable solid and determination of the limit load.

Third hypothesis. The opposite edges of a crack on its edges smoothly close, or stresses at the end (in the dead-end part) of cracks are finite.

This hypothesis, as already was noted, was first time advanced by S. A. Khristianovich [49]; it is mcre clearly formulated in work [5]; later in works [6, 13] its agreement with the general theorems of the mechanics of deformable bodies is shown. The hypothesis of the finiteness of stresses in the dead-end part of a crack essentially reflects the fact that any real body can sustain only finite tensile stresses.

In work [5] the accepted hypotheses essentially simplify analysis of the equilibrium state of a brittle body with macrocracks (especially from the point of view of determination of the limit loads for a preassigned configuration of cracks) and make it possible to formulate certain new positions of the theory of crack propagation.

For an example we examine a brittle (linear-elastic up to rupture) body weakened by plane cracks. Let us assume that such a body is loaded symmetrically relative to the plane of the cracks by a certain system of external stresses, including forces of interpartial cohesion G(x) in the dead-end part of the cracks. Then in the neighborhood of arbitrary point $M(l)$ (Fig. 5) of the contour of the crack the normal tensile stresses $\sigma_y(x)$ and vertical displacements v(x, 0) if its edges are expressed by formulas [8, 103]

$$\sigma_y(s_1) = \frac{N}{\sqrt{s_1}} + G(l) + 0(\sqrt{s_1}); \tag{11}$$

$$v(s_2) = \pm \frac{4(1-\nu^2)N}{E}\sqrt{s_2} + 0(s_2^{3/2}), \tag{12}$$

where N — coefficient of intensity of stresses in neighborhood of point $M(l)$ of contour of crack, which is a function of the effective loads, configuration of the body and form of the crack; s_1 and s_2 — small distances from contour of crack, shown on Fig. 5; $0(\sqrt{s_i}) \to 0$ when $s_i \to 0$ $(i = 1; 2)$.

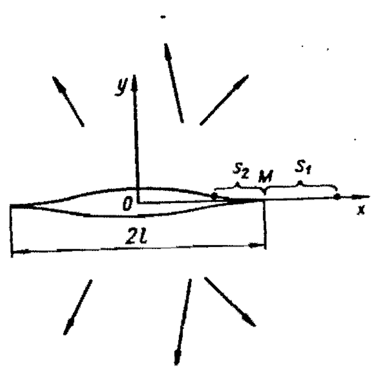


Fig. 5.

Inasmuch as the examined body is taken as linear-elastic up to rupture, the value of coefficient N can be represented in the form of the sum:

$$N = N_0 + N_m, \tag{13}$$

where N_0 – coefficient of intensity of tensile stresses in the neighborhood of the contour of the examined crack, calculated as a result of solving this problem as a problem of the classical theory of elasticity (neglecting forces of interpartial cohesion); N_m – coefficient of intensity of stress $\sigma_y(s_1)$ for such a body with a crack, but loaded only by forces of interpartial cohesion $G(x)$.

According to the third hypothesis tensile stresses $\sigma_y(s_1)$ in the dead-end part of the crack (i.e., when $s_1 \to 0$) must be limited, and the edges of the crack at $s_2 \to 0$ must smoothly close. From formulas (11) and (12) it is evident that this condition will be fulfilled only when the coefficient $N = 0$. Then expression (13) takes the form

$$N_0 + N_m = 0. \tag{14}$$

In accordance with the first hypothesis width d (see Fig. 4) of the end part of the crack in which cohesive forces $G(x)$ act, is small as compared to the whole region of the crack. Therefore the coefficient of intensity N_m of stresses $\sigma_y(s_1)$ can be determined using the formula of a plane theory of elasticity for an unbounded plane with a semi-infinite cut, when the edges of cut on sections $0 \leq t \leq d$ acts normal pressure $-G(t)$. In such a case (after calculation) we find

$$N_m = -\frac{1}{\pi}\int_0^d \frac{G(t)}{\sqrt{t}}\,dt. \tag{15}$$

According to the second hypothesis the distribution of cohesive forces $G(t)$ and width d of the end region in the neighborhood of points of contour of the crack, where cohesive force reach maximum, do not depend on the applied load, i.e., $d = d_* = \text{const}$. For such points the integral in formula (15) is a constant characterizing properties of the material under the assigned conditions. This

constant, designated in work by G. I. Barenblatt as K, is called the modulus of cohesion:

$$K = \int_0^{d_0} \frac{G(t)}{\sqrt{t}}\, dt. \tag{16}$$

Modulus of cohesion K is determined experimentally.

Thus, for points of the contour of a crack in which the highest possible intensity of cohesive forces is attained the following equality holds:

$$N_m = -\frac{1}{\pi} K, \tag{17}$$

and for points where the intensity of cohesive forces is less than the highest possible, we have the inequality

$$|N_m| < \frac{1}{\pi} K. \tag{18}$$

During monotonic growth of the load applied to a brittle body with a crack, forces of cohesion in the end region of the crack increase, ensuring the finiteness of the stresses and the smoothness of the closing of its edges on the contour of the crack. But this growth occurs as long as cohesive force do not attain the highest possible value at which equality (17) holds. Further increase of the external load (no matter how small) will lead to propagation of the crack (shift of its contour) in the neighborhood of points, where condition (17) holds, and points of the contour of the crack for which inequality (18) holds will remain fixed.

Thus, on the basis of equalities (14) and (17) for determination of external stresses, after the achievement of which a crack passes into the state of dynamic equilibrium we obtain the following formula:

$$N_0^* = \frac{1}{\pi} K. \tag{19}$$

where N_0^* is the maximum value of coefficient of intensity N_0 of elastic stresses on the contour of the crack.

As was noted above (see p. xxviii), a condition analogous to formula (19) is proposed in work by Irwin [195, 197] in a somewhat different interpretation (without analysis of the interpartial cohesive forces in the dead-end part of the crack.

By using condition (19), it is possible to formulate a convenient procedure for calculating values of the breaking load for a brittle body weakened by macrocracks. Actually, we assume that a brittle body weakened by cracks is subjected to a load by a certain system of monotonically increasing external loads Q, symmetric with respect to the plane of the cracks. Let us assume further that $\sigma_y(Q, a_i, s)$ are the elastic tensile stresses in the neighborhood of contour of one of the examined cracks, where s is a small distance between a certain point of the body located in the plane of the crack and the contour of the crack; a_i is the totality of parameters characterizing the dimensions of the contour of the crack.

Stresses $\sigma_y(Q, a_i, s)$ can be represented by analogy with formula (11) thusly:

$$\sigma_y(a_i, s) = \frac{N_0(Q, a_i)}{\sqrt{s}} + 0(1). \tag{20}$$

External load $Q = Q_*$ becomes maximum for a preassigned configuration of a crack if equality (19) holds at least in one point of its contour. For such points expression (20) can be written as:

$$\sigma_y(Q_*, a_i, s) = \frac{K}{\pi\sqrt{s}} + 0(1).$$

Using this equality and assuming that $s \to 0$, we obtain

$$\lim_{s\to 0}\{\sqrt{s}\,\sigma_y^*(Q_*, a_i, s)\} = \frac{K}{\pi}. \tag{21}$$

where $\sigma_y^*(Q_*, a_i, s)$ are the maximum tensile elastic stresses, determined within the bounds of the classical model of the theory of elasticity.

Such a force approach to determination of the limit load for a deformable body weakened by cracks does not contradict the power approach, but is more effective and permits solving a number of new problems of the theory of propagation of cracks [5, 8]. Let us show this by the example of a problem examined in Section 1 (see Fig. 1). In this case elastic tensile stresses $\sigma_\nu(x, 0)$ in the neighborhood of the end of the crack are calculated by formula (1). Substituting the value of these stresses into equation (21) and carrying out passage to the limit at $x \to l$ $(x \geq l, s = x - l)$, we find a solution to the problem:

$$p_* = \frac{\sqrt{2}K}{\pi\sqrt{l}}.$$

Since the force approach does not contradict the power approach [5], then, comparing the last equality with formulas (5), we obtain the following connection between modulus of cohesion K, elastic constants E, ν and density γ of the effective surface energy of the material correspondingly for the case of plane deformation and the plane generalized stressed state.

$$K = K_1 = \sqrt{\frac{\pi E\gamma}{1-\nu^2}} \text{ and } K = K_2 = \sqrt{\pi E\gamma}. \qquad (22)$$

Methods of experimental determination of values of the modulus of cohesion K or effective surface energy γ are described in works [5, 59, 129].

Footnotes

[1]The word brittle applies to a solid in which connection between stresses and deformations conforms to Hooke's law up to the instant when destruction begins.

[2]Detailed solution of this problem is given in Section 4 Chapter I.

[3]The load is the external force (stress), applied to a body.

[4]During the calculation of the function $W(l, p)$ Griffith used results of investigations by Inglis [194].

[5]Wolf's solution of the problem is approximate. This solution does not consider the contact stresses which appear between the edges of a crack when the strip is bent. A more complete problem, taking into account contact stresses, is given in work [122].

[6]Dynamic problems of fracture theory have been examined in recent years by many authors. A Bibliography of investigations in this region is contained in a work by G. I. Barenblatt [8].

[7]Work [216] uses results of works [157, 222] to determine values of $W(a, p)$. However, a general effective solution for the axisymmetrical problem of the theory of elasticity for a body with disk-like crack was given earlier by M. Ya. Leonov [63].

[8]Inaccuracies allowed by Frenkel' are noted in works [148] and [108].

[9]This problem is solved more simply in work [123].

[10]Elastic stresses in the neighborhood of the contour (ends) of a crack can be represented in the form $\frac{N}{\sqrt{s}}+0(1)$, where s is a small distance from the contour of the crack, N is the coefficient of intensity of stresses ages, 0 (1) is a bounded quantity when $s \to 0$.

CHAPTER I

SETTING UP THE PROBLEMS OF STATIC EQUILIBRIUM OF BRITTLE BODIES WEAKENED BY FRACTURES

1. Certain Initial Ideas and Definitions

In the mechanics of deformable solids we distinguish two categories of forces: external and internal forces. External forces (stresses) are different (external with respect to the solid) mechanical influences causing deformations of a body – relative change of the distance between its particles. Internal forces are the forces of interaction (cohesion) between the particles of the body, manifested as a result of its deformations.

Cohesive force, as it is known, essentially depend on the distance between the particles of a solid. The dependence of the intensity of these forces g on the distance r between two adjacent atomic planes in an ideal crystal is schematically depicted on Fig. 6. Function $g(r_0) = 0$, where r_0 is the normal interatomic distance, i.e., when the body is in the undeformed state.

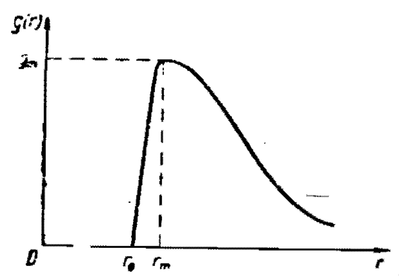


Fig. 6.

With an increase of distance r ($r > r_0$) cohesive forces g(r) rapidly increase, and at a certain value $r = r_m$ reach maximum $g(r_m) = g_m$; at a further increase of distance ($r > r_m$) these forces rapidly decrease. For amorphous bodies the dependence of forces of interpartial cohesion has the same qualitative character.

A detailed study of dependence g(r) is a very difficult problem, the complete solution of which has not been found even for one material. However, it is known [52, 164] that the area bounded by curve g(r) and axis Or when $r > r_m$ (see Fig. 6) is equal twice the density of surface energy γ_0 of a solid material. This energy numerically is equal to the work which it is necessary to expend on the formation of two new unit surfaces at an ideally brittle separation of body into parts:

$$2\gamma_0 = \int_{r_m}^{\infty} g(r)\,dr. \tag{I.1}$$

It is known [52, 164] also that when r changes within limits $r_0 \leq r \leq r_m$ the dependence of forces of interpartial cohesion approximately – accurate to small values $\left(\frac{r-r_0}{r_0}\right)^2$ – corresponds to Hooke's law (elastic region), i.e.,

$$g(r) = E\varepsilon, \tag{I.2}$$

where E – Young's modulus; $\varepsilon = \frac{r-r_0}{r_0}$ – elastic deformation (relative change of distance r).

The highest possible value of elastic deformation for an ideal crystal lattice is determined by the equality

$$\varepsilon_m = \frac{r_m - r_0}{r_0}.$$

Consequently, on the basis of equation (I.2) the maximum value of intensity of interpartial cohesion can be calculated by the formula

$$g_m \simeq E\varepsilon_m. \quad (I.3)$$

The maximum value of intensity of cohesive forces g_m determines the theoretical strength of a body, i.e., strength which a solid would have if its structure was an ideal crystal. An estimate of g_m, obtained within the bounds of the theory of an ideal cyrstal lattice [26, 52, 164], shows that values of the strength of crystal bodies are (0.01-0.1)E. Experiments of A. F. Ioffe and others [52] on the tensile strength of single crystals of rock salt when the surface of the samples is dissolved by water for the first time showed the reality of the theoretical strength, since in separate experiments a strength close to theoretical was found. The same conclusion was reached by experiments on the extension of filamentary crystals [105]. In particular, during the extension of filamentary crystals the maximum value of elastic deformation ε_m reaches a level of the order of 0.05, which in accordance with formula (I.3) gives a value of strength close to theoretical.

2. Calculation of the Technical Strength of Solids

The technical strength of real macroscopic solids is two to three orders lower than theoretical. As it is known this is because of the structural defects of the real macroscopic bodies.

Real solids, as a rule, are nonuniform. The structure of such bodies always has different defects: accumulation of dislocations or vacancies, block or grain boundaries, different inclusions, pores, crack and etc. The defects appear during the formation of the solid from the liquid state and in the process of its deformation. Due to heterogeneities of the material in regions where the defects are located, there is a concentration of internal stress, as a result of which in the separate regions of the body internal stresses reach the level of theoretical strength, although the mean values of stresses are comparatively small. This

means that in the overstrained regions of the deformable solid minute cracks can be formed (local destruction).

The appearance of the minute cracks in the volume of a deformable body still does not mean that its carrying capacity is completely exhausted. As experiments show, the resistance of such a body to the influence of an external load holds up to certain limits even after the minute cracks appear. From this, in particular, follows the fundamental scientific and practical value of a theory of the development of fracture defects in a deformable body, and also the development of methods of determining the resistivity of a material to the development of cracks.

According to the atomistic nature of real solids the determination of their strength requires that under a preassigned external influence the distribution of internal deformations and corresponding forces of interaction between particles of the body be studied. If the elementary (initial) particle of a real solid is assumed to be an atom, ion or molecule, such a study of the deformation and stresses is practically impossible.

In the mechanics of deformable solids the initial object of the study of stresses and deformation in a real solid is a small volume, but such that practically it can include many atoms (sometimes even many grains). In a mathematical relation this volume is assumed to be so small that the method of mathematical infinitesimal calculus will be used. Such a volume of a body usually is considered elastic, i.e., possessing a basic property of real solids, namely, the ability to be reversely deformed to a definite limit under the impact of external forces. It is assumed also that stress and deformation are continuous and differentiable functions of time and coordinates of a point of the body. In such a way the idea of a "continuous elastic medium" or an "elastic material continuum" is introduced in the mechanics of a deformable solid as a reference model of a real solid.

The transition from a discrete construction of solids to a model of an elastic material continuum means that the characteristics of strength and deformability of real solids is realized only through averaged values of internal stresses, and deformations on the areas of the volume elements.

In this case, i.e., within the bounds of the model of a "continuous elastic medium," the elastically deformed state of a solid is characterized by the fact that for every temperature of the body there exists a simple relationship between stresses and deformations in an arbitrary section (point) of the body. This dependence usually is considered linear and is expressed by Hooke's law. In the framework of such a model the effective methods of the mathematical theory of elasticity make it possible to determine the stresses and deformations in the solid if the external influences on the body do not cause deformations exceeding the elastic limit of the material [33, 90, 103, 140, 150, 157, 160].

However, the tensile properties of real solids, as mentioned above, essentially depend on the stressed and deformed state in those regions where deformations exceed the elastic limit. For example, when a solid is loaded with fractures in the neighborhood of the point of the crack deformations of the body always exceed the limit of elastic deformations. Consequently, the classical model of the theory of elasticity is inapplicable to such regions of a body. Because of this the development of elements of a quantitative theory of the strength of real solids, in particular resolution of problems about the propagation of cracks in a deformable brittle solid, requires that the model of the elastic medium must be supplemented by certain new properties, i.e., a new reference model must be created.

3. Reference Model of a Brittle Body

During the deformation of a solid weakened by cracks or other such defects in certain regions (for example, near the point of

the crack) appear deformations exceeding the elastic limit, i.e., in the body appear layers (regions) in which Hooke's law does not hold. In brittle materials such regions are small volumes (interlayers) as compared to that part of the body which is deformed elastically. Considering this and based on conclusions of the preceding section, a true solid deformed by a system of external forces will be examined as a certain solid elastic body (material continuum) deformed elastically everywhere, except certain layers of material in which deformation of the body exceed the elastic limit. Such an "overstrained" layer of a body, where material is deformed beyond the limits of elasticity, can be mentally removed from the body, forming thus in the body certain slots, cracks to whose surfaces are applied stresses corresponding to the action of the remote material.

If in a real solid there are initial fracture defects (narrow slots), in an elastic model solid such defects can be represented in the form of cuts whose opposite edges do not interact.[1]

Thus, the problem about the stressed and deformed state of a solid, when in body are initial cracks, and also a layer of "overstrained" material, can be reduced to the problem of the stressed and deformed state in an ideally elastic body weakened by cracks whose surfaces interact according to a particular law. If the interaction forces between the edges of cracks are determined, the problem leads to a certain mixed problem of the mathematical theory of elasticity.

With such an approach to the problem first of all it is necessary to clarify the relationship between forces and deformations in that part of a deformable solid where deformations exceed the elastic limit. In general the solution of such a problem runs into great difficulties. However during investigation of the problems about the maximum equilibrium state of brittle bodies weakened by cracks the approximate determination of the forces of interaction between edges of a crack is possible.

Indeed, let us examine a brittle body weakened by a fracture. Let us assume that such body is subjected to extension by a system of external forces. Then for any small but nonzero external load in the neighborhood of the point of the crack will appear deformations, exceeding the elastic limit. Such regions of the body can be examined as minute cracks, i.e., cracks whose opposite edges interact. Since the examined body is brittle, forces of interaction (attraction between edges of minute cracks) are determined by the intensity of interpartial cohesive forces $g(r)$ at $r > r_m$ (see Fig. 6), where $r-r_m$ is the distance between the edges of the cracks.

Inasmuch as forces $g(r)$ at $r > r_m$ have been incompletely studied, then, based on the above general properties of function $g(r)$ and considering a macroscopic elastic model we will introduce the following hypothesis.

Basic hypothesis. For an ideally brittle body within the bounds of the model of a continuous medium the forces of weakened interpartial connections ($g(r)$ at $r > r_m$) are determined so: if the distance between edges of the crack does not exceed a certain value δ_k (constant for a given material under preassigned conditions), they are equal to the constant σ_0; if, however, the distance between the edges of a crack exceeds δ_k, the cohesive forces between them are zero (Fig. 7).

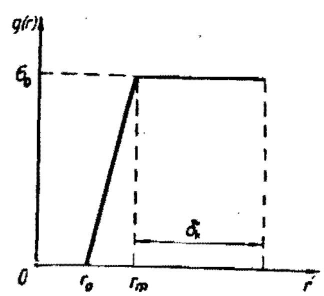


Fig. 7.

According to formula (I.1) values of σ_0, δ_k and γ are interconnected by the equality

$$2\gamma = \sigma_0 \delta_k, \tag{I.4}$$

The quantity σ_0 is equal to the limit of brittle strength for a given material [16, 158]. Subsequently it will be shown that in the solution of problems of the theory of macrocracks, i.e., during the investigation of propagation of initial cracks, whose characteristic linear dimensions are rather large, not the value of σ_0 is essential, but the value of the product $\sigma_0\delta_k = 2\gamma$.

Thus, the reference model of a real brittle body constitutes a solid elastic body which is characterized by the following properties:

1) maximum tensile stresses appearing in such a body do not exceed value σ_0 – the limit brittle strength of the material;

2) the relationship between stresses and deformations is expressed by Hooke's law if tensile stresses do not reach σ_0;

3) in the body microscopic crack will be formed (region of weakened interpartial connections) if the maximum tensile stresses calculated on the basis of the linear theory of elasticity reach σ_0;

4) opposite edges of these microscopic cracks are attracted with a stress σ_0 if the distance between the edges of the microscopic cracks do not exceed δ_k, and do not interact if this distance exceeds δ_k. For a brittle material, for example for glass parameter δ_k is determined from the conditions of equality of values of the density of effective surface energy of the examined real body and the surface energy of the model formulated above model, i.e., from equality (I.4). This model is formulated in works [66, 70, 71, 112].

The above reference model of a brittle body with cracks (δ_k-model) is determined by yet two characteristics besides the elastic coefficients (Hooke's law): limit of brittle strength σ_0 and critical interval δ_k. Within the bounds of this model it is

possible to formulate a general problem about the appearance and development (propagation) of cracks in a real brittle body as a corresponding power problem of the statics of a deformable elastic body.

In the formulation of such a problem we note the following. A fracture (slot) in a real solid within the bounds of the formulated model is represented as a cut. The region of the crack where the distance between its edges exceeds δ_k, i.e., the region of torn interpartial connections, corresponds to sections of the cut whose opposite edges do not interact with each other. The dead-end sections of the crack, i.e., region of the body where the material is deformed beyond the limits of elasticity (region of weakened interpartial connections) correspond in the δ_k-model to cuts whose opposite edges are attracted with a stress of σ_0 while the distance between them is less than δ_k.

Consequently, the propagation of a real crack (development of brittle rupture) in a δ_k-model is defined as the process of transition of points of a region of weakened interpartial connections into a region of torn connections. Consequently, the condition of propagation of a real crack in a deformable brittle body can be written as (see Fig. 8):

$$2v_n(l_0, l, q_*) = \delta_k, \tag{I.5}$$

where $v_n(l_0, l, q)$ is the normal vectorial component of displacements of points of the edges of the crack, calculated by methods of the linear theory of elasticity within the bounds of the formulated model when $q = q_*$; l_0 – characteristic linear dimension of region of initial crack; q_* – parameter characterizing external load, $q_* = q$ (on Fig. 8 $l-l_0$ – characteristic linear dimension of dead-end part of crack).

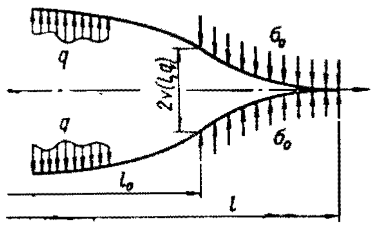


Fig. 8.

The least value of the external load $q = q_*$ for a brittle body with ruptures, upon reaching which condition (I.5) is realized and, consequently, the possibility of crack propagation, is called the limit load. The initial crack for which condition (I.5) holds at least in one point of its dead-end part, is called the maximum equilibrium crack.

Thus, study of crack propagation in a deformable brittle solid under a monotonically increasing external load in this case leads to the solution of a problem of linear theory of elasticity within the bounds of the formulated reference model and determination of parameters of the external load satisfying condition (I.5).

4. Generalized Griffith Problem

We will examine within the bounds of the δ_k-model of a brittle body the problem of maximum equilibrium of an unbounded plate weakened by isolated rectilinear crack $2l_0$ (Fig. 9), not introducing limitations on the length of the examined crack.

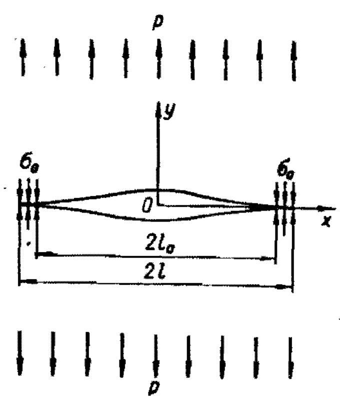


Fig. 9.

Let us assume that in points at infinity of such a plate are applied external tensile stresses p, directed perpendicular to the line of the crack. Let us determine the limit value of stresses $p = p_*$, i.e., the value after which the crack will cross into the transient-equilibrium state and it will be possible for it to propagate across the cross section of the plate. Furthermore, we will investigate the structure of the dead-end part of the crack.

Let us refer the examined plate to a rectangular system of cartesian coordinates xOy, assuming that the thickness of plate is equal to unity, and the crack is located along axis x on segment $|x| \leq l_0$ (Fig. 9). Let us note that no matter how small are stresses p in the neighborhood of the ends of a real crack stresses $\sigma_y(x, 0)$, determined according to the linear theory of elasticity, exceed the limit of brittle strength of the material, i.e., in this part of the body appear regions of weakened bonds. On the basis of the symmetry of the problem on properties of a δ_k-model of a brittle body these regions can be examined as cuts along the x-axis when $l_0 \leq |x| \leq l$, the opposite sides of which are attracted with stress σ_0. The value of parameter l is still unknown.

Thus, the problem about the stressed and deformed state in plate with a crack reduces to the following problem of the mathematical theory of elasticity. In elastic plane xOy (see Fig. 9) is a cut of length $2l$ ($-l \leq x \leq l$). On the surfaces of this cut act stresses

$$\tau_{xy}(x, 0) = 0; \quad \sigma_y(x, 0) = \begin{cases} 0 & \text{when } |x| \leqslant l_0; \\ \sigma_0 & \text{when } l_0 < |x| \leqslant l, \end{cases} \tag{I.6}$$

and in points at infinity of the plane

$$\sigma_y(x, \infty) = p. \tag{I.6a}$$

If from the stressed state appearing in the elastic plane with cut $2l$ under boundary conditions (I.6) and (I.6a) we subtract

uniform stressed state $\sigma_x = \tau_{xy} = 0$ and $\sigma_y(x, y) = p$, then we obtain a certain supplementing auxiliary) stressed state, vanishing at infinity. On the surface of the cut this state is determined by boundary conditions

$$\tau_{xy}(x, 0) = 0; \quad q_n(x) = \begin{cases} p & \text{when } |x| \leqslant l_0; \\ p - \sigma_0 & \text{wher } l_0 < |x| \leqslant l, \end{cases} \tag{I.7}$$

where $q_n(x) = -\sigma_y(x, 0)$ is the normal pressure on the surfaces of the cut for the auxiliary problem.

A uniform stressed state does not depend on the dimensions of the crack, therefore during the investigation of the development of a crack the supplementing (auxiliary) state of strain[2] essential.

Considering this, we will determine vertical displacements $v(x, 0)$ when $|x| \leq l$ and stresses $\sigma_y(x, 0)$ when $|x| \geq l$ for the examined plate with a crack, when on the contour of the crack boundary conditions (I.7) are assigned. For this we will use results of work [70], where it is shown that vertical displacements $v(x, 0)$ of the sides of a cut of length $2l$ in elastic unbounded plane xOy (Fig. 10), when at points $(\xi, +0)$ and $(\xi, -0)$ of the cut normal concentrated forces P are applied, equal in magnitude and opposite in direction, are expressed by the formula

$$\begin{gathered} v_0(x, 0) = -cP\Gamma(l, x, \xi), \\ \Gamma(l, x, \xi) = \ln \frac{l^2 - x\xi - \sqrt{(l^2 - x^2)(l^2 - \xi^2)}}{l^2 - x\xi + \sqrt{(l^2 - x^2)(l^2 - \xi^2)}} \quad (|x| \leqslant l). \end{gathered} \tag{I.8}$$

Here c is a constant which for generalized plane stress is equal to $1/\pi E$, and for plane deformation $\frac{(1-\nu^2)}{\pi E}$.

Stress $\sigma_y^0(x, 0)$ when $|x| \geq l$ is determined as:

$$\sigma_y^0(x, 0) = \frac{P\sqrt{l^2 - \xi^2}}{\pi \sqrt{x^2 - l^2}\,|x - \xi|}. \tag{I.9}$$

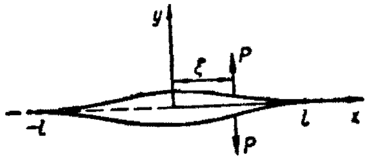


Fig. 10.

If to the sides of a cut of length $2l$ is applied normal pressure $q_n(x)$, represented by equality (I.7), then in accordance with expressions (I.8) and (I.9) vertical displacements $v(x, 0)$ of the sides of this cut when $|x| \leq l$ and stresses $\sigma_y(x, 0)$ when $x > l$ are determined by the formulas

$$v(x, 0) = -c\int_{-l}^{l} q_n(\xi)\,\Gamma(l, x, \xi)\,d\xi \quad (|x| \leqslant l); \tag{I.10}$$

$$\sigma_y(x, 0) = \frac{1}{\pi\sqrt{x^2 - l^2}}\int_{-l}^{l} \frac{q_n(\xi)\sqrt{l^2-\xi^2}}{x-\xi}\,d\xi \quad (x \geqslant l). \tag{I.11}$$

Substituting in formulas (I.10) and (I.11) the expression for $q_n(\xi)$ from equality (I.7) and calculating the necessary integrals we will obtain

$$v(x, 0) = 2\pi cp\sqrt{l^2 - x^2} + c\sigma_0\Big\{(x - l_0)\,\Gamma(l, x, l_0) - \\ -(x + l_0)\,\Gamma(l, x, -l_0) - 4\sqrt{l^2 - x^2}\arccos\frac{l_0}{l}\Big\} \quad (|x| \leqslant l); \tag{I.12}$$

$$\sigma_y(x, 0) = \frac{1}{\pi\sqrt{x^2 - l^2}}\Big\{\pi(p - \sigma_0)\left(x - \sqrt{x^2 - l^2}\right) + \\ + 2\sigma_0 x \arcsin\frac{l_0}{l} + \sigma_0\sqrt{x^2 - l^2}\,\times \\ \times\left[\arcsin\frac{l^2 - xl_0}{l(x - l_0)} - \arcsin\frac{l^2 + xl_0}{l(x + l_0)}\right]\Big\} \quad (x \geqslant l). \tag{I.13}$$

Stresses $\sigma_y(x, 0)$, represented by formula (I.13) for an arbitrary value of parameter l become unlimited if $x \to l$. However, considering properties of the reference model, these stresses can be limited. Consequently, parameter l should be such that the

condition of finiteness of stresses $\sigma_y(x, 0)$ holds when $x \to l$. A necessary (and also sufficient) condition for the finiteness of stresses $\sigma_y(x, 0)$ when $x \to l$, as follows from formula (I.13), is that the expression in the braces of this formula is equal to zero. It is considered that $x = l$, and equating the expression obtained in this case in the braces to zero, we obtain the following formula for determination of parameter l:

$$p = \frac{2}{\pi}\sigma_0 \arccos\frac{l_0}{l} \quad \text{when} \quad l = l_0 \sec\frac{\pi p}{2\sigma_0}. \tag{I.14}$$

Using equality (I.14), formula (I.12) can be transformed to such a form:

$$v(x, 0) = c\sigma_0 \{(x - l_0)\Gamma(l, x, l_0) - (x + l_0)\Gamma(l, x, -l_0)\}, \tag{I.15}$$

where $-l \leq x \leq l$.

Differentiating equality (I.15) with respect to x and taking into account that

$$\frac{\partial\Gamma(l, x, \xi)}{\partial x} = \frac{2\sqrt{l^2 - \xi^2}}{\sqrt{l^2 - x^2}\,(x - \xi)}, \tag{I.16}$$

we obtain

$$v'(x, 0) = c\sigma_0 \{\Gamma(l, x, l_0) + \Gamma(l, x, -l_0)\}. \tag{I.17}$$

From equations (I.15) and (I.17) it follows that:

$$v(l, 0) = v'_x(l, 0) = 0 \quad \left(v'_x = \frac{\partial v}{\partial x}\right). \tag{I.18}$$

Equalities (I.18) show that (in contrast to the ideas of Griffith on the roundness of the ends of a developed crack [190]) the opposite sides of a real equilibrium crack on its contour is closed smoothly (with zero angle of opening).

Using formulas (I.14) and (I.15), it is easy to determine the distance $2v(x, 0)$ between the opposite sides of an equilibrium crack when $|x| \leq l$ for any value of external load $p \leq \sigma_0$. In particular, the distance between the opposite sides of a crack in points with abscissa $x = \pm l_0$ are expressed so:

$$2v(\pm l_0, 0) = -8 l_0 c \sigma_0 \ln\left(\cos\frac{\pi p}{2\sigma_0}\right). \qquad (I.19)$$

In accordance with the accepted model the limit value of the external load for a body with a crack is determined from equality (I.5). Consequently, to determine the limit load $p = p_*$ at an assigned length l_0 of the initial crack (see Fig. 9) on the basis of equality (I.19) and condition (I.5) we obtain the following formula:

$$\delta_k = 2v(\pm l_0, 0) = -8 l_0 c \sigma_0 \ln\left(\cos\frac{\pi p_*}{2\sigma_0}\right)$$

or

$$p_* = \frac{2}{\pi}\sigma_0 \arccos\left\{e^{-\frac{\delta_k}{8 l_0 \sigma_0 c}}\right\}. \qquad (I.20)$$

Formula (I.20) is obtained in work [112], and also by other means in work [32].

If the initial length of a crack l_0 is large enough to consider $l_0 >> \delta_k$, then, keeping in formula (I.20) the magnitudes of only the first order of smallness and taking into account equality (I.4), we obtain the Griffith formula:

$$p_* = \sqrt{\frac{2E\gamma}{\pi(1-\nu^2) l_0}}. \qquad (I.21)$$

When $l_0 \to 0$ formula (I.21) gives an infinitely large value of p_*. From formula (I.20) when $l_0 \to 0$ we have $p_* \to \sigma_0$, i.e., a plate with a crack of "zero length" has the strength of a faultless

plate. This physically trivial result does not follow from the present generalizations of the theory of Griffith.

5. Structure of the Edge of an Equilibrium Crack

Let us examine more specifically the structure of the edge of an equilibrium crack in a brittle plate (see Fig. 9). This question, as already was noted, has a fundamental value; in particular, it is very important during an investigation of the influence of surface active media on deformation and rupture of solids [81, 82, 147].

Formulas (I.8) and (I.9) permit representing displacements v(x, 0) of the sides of a crack and stress $\sigma_y(x, 0)$ outside this crack for any normal load applied to the sides of the crack. Having this in mind, we will assume that to the sides of the crack (Fig. 11) is applied arbitrary normal pressure $q_0(x)$, and in the dead-end part of the crack ($l_0 < |x| \leq l$) act weakened forces of molecular cohesion of the general form G(x), which characterize the state of strain of a material deformed beyond the limits of elasticity. Thus, to the sides of the cut will be applied normal pressure

$$q_n(x) = \begin{cases} q_0(x) & \text{when } |x| \leqslant l_0; \\ q_0(x) - G(x) & \text{when } l_0 < |x| \leqslant l. \end{cases} \quad \text{(I.22)}$$

Then, according to (I.6), (I.8) and (I.9), we obtain

$$v(x, 0) = -c\int_{-l}^{l} q_n(\xi)\,\Gamma(l, x, \xi)\,d\xi \quad (|x| \leqslant l); \quad \text{(I.23)}$$

$$\sigma_y(x, 0) = \frac{1}{\pi\sqrt{x^2 - l^2}}\int_{-l}^{l} \frac{q_n(\xi)\sqrt{l^2 - \xi^2}\,d\xi}{x - \xi} \quad (x > l). \quad \text{(I.24)}$$

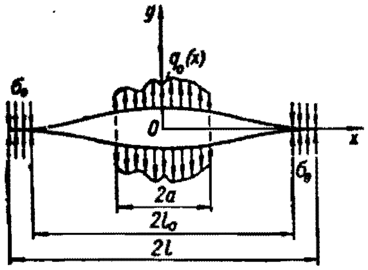


Fig. 11.

Inasmuch as within the framework of the accepted model in any point of a body tensile stresses $\sigma_y(x, 0)$ must not exceed the level of brittle strength, then as follows from expression (I.24), the following condition should hold:

$$\lim_{x \to l+0} \int_{-l}^{l} \frac{q_n(\xi)\sqrt{l^2-\xi^2}\,d\xi}{x-\xi} = 0. \qquad (I.25)$$

For sufficiently general assumptions with respect to function $q_n(\xi)$ the integral in formula (I.24) can be represented in the following form:

$$\int_{-l}^{l} \frac{q_n(\xi)\sqrt{l^2-\xi^2}}{x-\xi}\,d\xi = \pi q_n(x)\sqrt{x^2-l^2} + \pi i A_\infty(x) \quad (x > l), \qquad (I.26)$$

where $i = \sqrt{-1}$; $A_\infty(x)$ – the main part [103] of function $q_n(z)\sqrt{l^2-z^2}$ as $|z| \to \infty$, when $z = x + iy$, $y \to 0$.

According to formula (I.26) condition (I.25) will hold if

$$A_\infty(l) = 0. \qquad (I.27)$$

In this case stresses $\sigma_y(x, 0)$ determined by formula (I.24) will be not only limited (equal to $q_n(l)$), but also continuous at a point $x = l$.

Differentiating formula (I.23) with respect to x, we obtain

$$v_x'(x, 0) = \frac{-2c}{\sqrt{l^2-x^2}} \int_{-l}^{l} \frac{q_n(\xi)\sqrt{l^2-\xi^2}\,d\xi}{x-\xi} \quad (|x| \leq l). \qquad (I.28)$$

The integral in the formula, in the same way as (I.26) is determined by the equality

$$\int_{-l}^{l} \frac{q_n(\xi)\sqrt{l^2-\xi^2}}{x-\xi}\, d\xi = \pi i A_\infty(x) \quad (|x| \leqslant l), \tag{I.29}$$

where $A_\infty(x)$ has the same value as in formula (I.26).

From formulas (I.27)-(I.29) it follows that when $x \to l$ we have

$$\lim_{x \to l} v'_x(x, 0) = 0. \tag{I.30}$$

Thus, according to (I.30) it is possible to establish a general property of the structure of the edge of an equilibrium crack: the opposite sides of an equilibrium crack on its contour are closed smoothly (with zero angle of opening). Furthermore, equalities (I.25) and (I.30) show that the condition of the evenness of closing of the sides of a crack on its contour is equivalent to the condition of finiteness of tensile stresses in these points.[3]

Let us check the last affirmation by the example of the generalized Griffith problem. For this we will differentiate formula (I.12) with respect to x and will equate to zero the value of v'_x (x, 0) at $x = l$. As a result we obtain the equality

$$l = \frac{l_0}{\cos\frac{\pi p}{2\sigma_0}}. \tag{I.31}$$

Using equality (I.31), formula (I.12) is easily transformed to the form

$$v(x, 0) = c\sigma_0 \{(x - l_0)\Gamma(l, x, l_0) - (x + l_0)\Gamma(l, x, -l_0)\}. \tag{I.32}$$

Here

$$-l \leqslant x \leqslant l;$$

$$\Gamma(l, x, \xi_0) = \ln \frac{l^2 - x\xi_0 - \sqrt{(l^2-x^2)(l^2-\xi^2)}}{l^2 - x\xi_0 + \sqrt{(l^2-x^2)(l^2-\xi^2)}},$$

where $\xi_0 = \pm l_0$.

Thus, formula (I.32) coincides with formula (I.15).

In the example of the generalized Griffith problem let us investigate the structure of the edge of an equilibrium crack on the boundary between the region of weakened and disturbed interpartial connections, i.e., in the neighborhood of points $x = \pm l_0$ (see Fig. 9). For this let us note that $v(x, 0)$ when $x = \pm l_0$ is a limited function, and the derivative of this function with respect to x is expressed by equality (I.19). According to the properties of function $\Gamma(l, x, \xi_0)$ where $\xi_0 = \pm l_0$, we have

$$\lim_{x \to l_0 \pm 0} v'_x(x, 0) \to \infty, \tag{I.33}$$

and the second derivative of function $v(x, 0)$ changes sign when it crosses through point $x = \pm l_0$. This means that points $(+l_0, 0)$ and $(-l_0, 0)$ are points of inflection of this function. Function $v(x, 0)$, represented by formula (I.32), in the interval $-l \leq x \leq l$ has no other singular points. The boundary between the region of torn interpartial connections (region of initial crack) and the region of weakened interpartial connections (dead-end part of crack) is characterized within the bounds of the accepted δ_k-model by the presence of singular points (points of inflection) in the structure of the edge of the equilibrium crack.

To illustrate the change of structure of the edge of the equilibrium crack in a plate (see Fig. 9) when external stress $p = kp_*$ monotonically increases, we will examine the following example. Let us assume that $l_0 = 0.1$ cm; $E = 10^7$ N/cm^2; $\gamma = 2 \cdot 10^{-4}$ J/cm^2; $\sigma_0 = 0.01E$. Then, using formula (I.20) and equality $\sigma_0 \delta_k = 2\gamma$, we easily find $p_* = 0.0113945\sigma_0$.

Hence by formula (I.32) for every value of load $p = kp_*$ we obtain

the value of parameter $l = l_0(1 + 10^{-5}\delta_1)$, where δ_1 is found from the data below:

k	0,1	0,5	0,8	1,0
δ_1 . . .	0,16	4,00	10,25	16,02

For convenience of calculations formula (I.32) is written in the following form:

$$V\left(\frac{x}{l_0}\right) = \frac{v(x, 0)}{c\sigma_0 l_0} = \left(\frac{x}{l} - 1\right)\Gamma(l, x, l_0) - \left(\frac{x}{l_0} + 1\right)\Gamma(l, x - l_0).$$

Using this formula and the above data it is possible to calculate the value of function $v(\frac{x}{l_0})$ for every point x in the interval $|x| \leq l$. On Fig. 12a according to Table 2 are built graphs characterizing the change of form (structure) of the edge of the equilibrium crack during monotonic growth of load $p = kp_*$ (Fig. 12b shows the graph of the opening of only the dead-end part of the crack for the examined example).

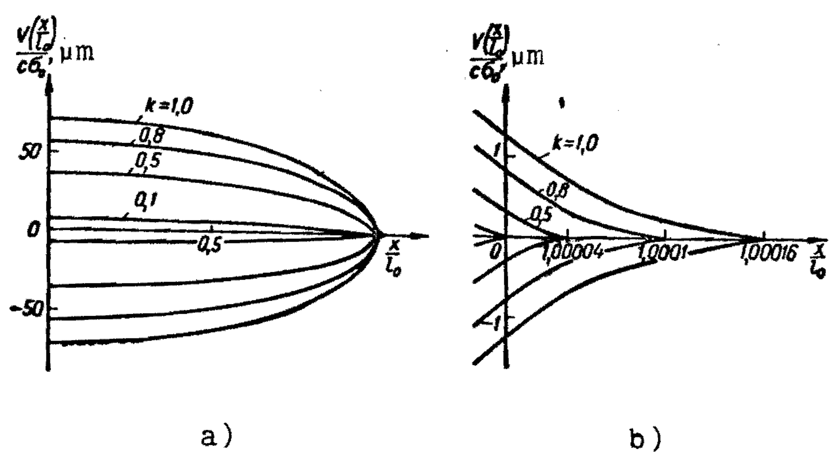


Fig. 12.

Table 2.

$\frac{x}{l_*}$	$V\left(\frac{x}{l_*}\right) l_* \cdot 10^3$, мм $k = 0{,}1$; $\delta_1 = 0{,}16$	$k = 0{,}5$; $\delta_1 = 4{,}00$	$k = 0{,}8$; $\delta_1 = 10{,}25$	$k = 1{,}0$; $\delta_1 = 16{,}02$
0,00	7,15538	35,77656	57,28336	71,61830
0,25	6,92818	34,64057	55,46417	69,34571
0,50	6,19674	30,98359	49,60928	62,02333
0,80	4,29332	21,46643	34,37182	42,97497
0,90	3,11896	15,59554	24,97282	31,22361
0,95	2,23428	11,17261	17,89237	22,37160
0,99996	0,064636	0,367284	0,731400	0,947328
1,00000	0,006400	0,159998	0,410060	0,640750
1,0000008	0,001702	—	—	—
1,00002	—	0,042624	—	—
1,00003	—	0,014081	0,197925	—
1,00004	—	—	—	0,344235
1,00005	—	—	0,113738	—
1,00008	—	—	—	0,171058

6. Basic Positions of the Theory of Macroscopic Cracks

If the characteristic linear dimension of an initial crack D (Fig. 13, where plane of drawing is the plane of the crack) is rather large as compared to parameter δ_k of the given material, then the characteristic linear dimension of the region of weakened interpartial connections (linear dimension of dead-end part of crack d) is small as compared to D:

$$\delta_k \ll D, \; d \ll D. \tag{I.34}$$

Initial cracks, for which are inequalities (I.34) are valid, we will call macroscopic. By this definition macroscopic cracks are a class of cracks obeying the first hypothesis of the theory of G. I. Barenblatt [8] (see Section 4).

The minimum characteristic linear dimension of a macroscopic crack can be estimated by the example of the generalized Griffith problem. For this formula (I.20) will be written in the following form:

$$p_*^{(1)} = \frac{2}{\pi}\sigma_0 \arccos\left(e^{-\frac{d_*}{l_*}}\right) \quad \left(d_* = \frac{\delta_k}{8c\sigma_*}\right). \tag{I.35}$$

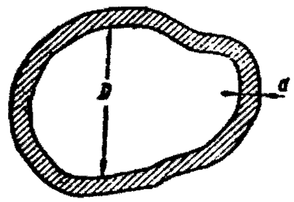


Fig. 13.

For a macroscopic crack inequalities (I.34) are valid. Therefore, according to formula (I.35), accurate to small values of d_*/l_0 we have

$$p_*^{*} = \frac{2}{\pi}\sigma_0\sqrt{\frac{2d_*}{l_*}}. \qquad (I.36)$$

On Fig. 14 are graphs of the change of breaking points p_* depending upon $\lambda = l_0/d_*$, where curve 1 is built according to data from formula (I.35), and curve 2 according to formula (I.36). According to this graph, beginning at $\lambda = 10$, curves of the change of limit load p_* practically coincide, since the difference between them does not exceed 2% (Table 3). It follows from this that the use of formula (I.36) to determine the limit load can be considered as justified if $l_0 > 10d_*$.

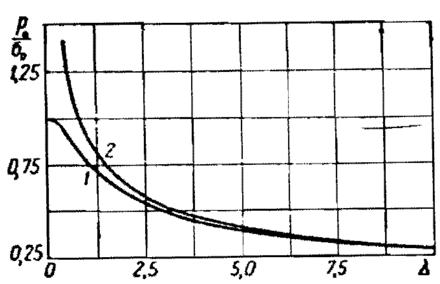


Fig. 14.

Thus, we arrive at the following rough estimate of the characteristic linear dimension D of an initial macroscopic crack:

$$D \geqslant 10d_* \simeq \frac{20\gamma}{8c\sigma_0^2}. \qquad (I.37)$$

Table 3.

$\frac{l_*}{d_*}$	$\frac{p_*^{(2)}}{\sigma_*}$	$\frac{p_*^{(1)}}{\sigma_*}$	$\frac{p_*^{(2)}-p_*^{(1)}}{p_*^{(1)}}$
0,00	—	1,000000	—
0,05	4,026337	1,000000	3,0263
0,10	2,847050	0,999971	1,8471
0,20	2,013168	0,995710	1,0218
0,40	1,423525	0,947684	0,5021
0,60	1,162303	0,879031	0,3222
0,80	1,006584	0,815013	0,2350
1,00	0,900316	0,760168	0,1394
2,00	0,636620	0,585121	0,0880
3,00	0,519798	0,491457	0,0577
4,00	0,450158	0,431656	0,0428
5,00	0,402634	0,389356	0,0341
6,00	0,367552	0,357433	0,0283
7,00	0,340287	0,332246	0,0242
8,00	0,318310	0,311722	0,0211
9,00	0,300105	0,294580	0,0187
10,00	0,284705	0,279984	0,0168
50,00	0,127324	0,126900	0,0033
100,00	0,090032	0,089881	0,0017

For macroscopic cracks we will easily obtain relationships shown in Section 4 of the introduction according to the δ_k-model of a brittle body. Let us consider this in an example.

Let us assume that in a plate is a linear isolated macrocrack of length $2l_0$. We introduce rectilinear system of cartesian coordinates x0y as is shown on Fig. 11, and assume that to the sides of the crack is applied normal pressure $q_0(x)$ when $|x| \leq d$ $(d \leq l_0)$, and in points at infinity of the plate there are no stresses. By the properties of the reference model and in the same way as in Sections 4 and 5 vertical displacements v(x, 0) of the sides of the crack for the examined problem are represented in the form:

$$v(x, 0) = -c\int_{-l}^{l} q_n(\xi)\,\Gamma(l, x, \xi)\,d\xi \quad (|x| \leqslant l), \tag{I.38}$$

where l is the abscissa of the boundary of the contour of the crack.

Normal pressure $q_n(\xi)$ in formula (I.38) is determined by equality

$$q_n(\xi) = \begin{cases} q_0(\xi) & \text{when } |\xi| \leqslant a; \\ 0 & \text{when } a < |\xi| \leqslant l_0; \\ -\sigma_0 & \text{when } l_0 < |x| \leqslant l. \end{cases} \tag{I.39}$$

For our case

$$\sigma_y(x, 0) = \frac{1}{\pi\sqrt{x^2 - l^2}} \int_{-l}^{l} \frac{q_n(\xi)\sqrt{l^2-\xi^2}}{x-\xi}\, d\xi \quad (x > l). \tag{I.40}$$

Due to the limitedness of stresses $\sigma_y(x, 0)$ when $x \to l + 0$ the following equality should hold:

$$\lim_{x \to l+0} \int_{-l}^{l} \frac{q_n(\xi)\sqrt{l^2-\xi^2}}{x-\xi}\, d\xi = 0. \tag{I.41}$$

Using expression (I.39), equality (I.41) can be written as:

$$\lim_{x \to l+0} \sigma_0 \left\{ \int_{-l}^{-l_0} \frac{\sqrt{l^2-\xi^2}}{x-\xi}\, d\xi + \int_{l_0}^{l} \frac{\sqrt{l^2-\xi^2}}{x-\xi}\, d\xi \right\} = \lim_{x \to l+0} \int_{-a}^{a} \frac{q_0(\xi)\sqrt{l^2-\xi^2}}{x-\xi}\, d\xi. \tag{I.41a}$$

Let us introduce parameter

$$\varepsilon = \frac{l - l_0}{l_0}. \tag{I.42}$$

Our crack is macroscopic, therefore $(l - l_0) \ll l_0$ $(l + l_0 \approx 2l_0 \approx 2l)$ and, consequently $\varepsilon \leq 1$. Having this in mind, equality (I.41a) accurate to a magnitude of the order of ε can be represented in the form

$$2l_0\sigma_0\sqrt{2\varepsilon} = \int_{-a}^{a} \frac{q_0(\xi)\sqrt{l^2-\xi^2}}{l-\xi}\, d\xi. \tag{I.43}$$

Just as before, from formula (I.38) accurate to small values of ε we obtain

$$v(l_0, 0) = -4cl_0\sigma_0\varepsilon + 2c\sqrt{2\varepsilon} \int_{-a}^{a} \frac{q_0(\xi)\sqrt{l^2-\xi^2}}{l-\xi}\, d\xi. \tag{I.44}$$

If parameter determining external load $q_0(x)$, are such that the examined crack becomes maximum equilibrium, then condition (1.5) holds, i.e., $v(l_0, 0) = \frac{1}{2}\delta_k$. In this case ε reaches a certain limit value $\varepsilon_* = \frac{l_* - l_0}{l_0}$ and on the basis of equality (I.5), (I.43) and (I.44) is expressed by formula

$$\varepsilon_* = \frac{\delta_k}{8cl_0\sigma_0}. \qquad (I.45)$$

Hence and on the basis of equality (10) we obtain

$$l_* - l_0 = \frac{\delta_k}{8c\sigma_0}, \qquad (I.46)$$

where l_* is the value of parameter l for a maximum equilibrium crack.

In the right part of formula (I.46) are contained values independent of load and initial dimension of the crack, therefore we can conclude that for a given material under preassigned conditions of the dead-end region of a maximum equilibrium macroscopic crack is a constant. Thus, for a macroscopic cracks the hypothesis on the autonomy of the end region of a maximum equilibrium crack is valid.

Within the bounds of the formulated model of a brittle body in the case of macroscopic cracks the following formula for determination of limit load is valid:

$$\lim_{s \to 0} [\sqrt{s}\sigma_y(s, q_*)] = \frac{K}{\pi}, \qquad (I.47)$$

where $\sigma_y(s, q_*)$ are the rupture stresses calculated for the examined body on the basis of the classical model of the theory of elasticity during loading $q \leq q_*$; s – distance of points of the body located in the plane of the crack, from the contour of the crack; K – modulus of cohesion [8].

Carrying out passage to the limit in the left part of equality (I.41a) and calculating integrals in this part, this equality can be represented accurate to small values (of the order of ε inclusively) in the following form:

$$\frac{2l_0\sigma_0\sqrt{2\varepsilon}}{\pi}=\lim_{x\to l_0+0}\{\sqrt{x^2-l_0^2}\,\sigma_y(x,q)\}\quad (x\geqslant l_0), \qquad (I.48)$$

where $\sigma_y(x, q)$ are the elastic rupture stresses for a plate with a macroscopic crack of length $2l_0$, when to the sides of this crack when $|x| \leq a$ are applied external loads $q_0(x)$,

$$\sigma_y(x,q)=\frac{1}{\pi\sqrt{x^2-l_0^2}}\int_{-a}^{a}\frac{q_0(\xi)\sqrt{l_0^2-\xi^2}}{x-\xi}\,d\xi.$$

Moreover

$$\sqrt{x^2-l_0^2}=\sqrt{s}\sqrt{2l_0+s}, \qquad (I.49)$$

where

$$s=(x-l_0).$$

Formula (I.48) is valid for any load not exceeding the limit value. If, however, the external load reaches the limit value (in our case $q = q_*$), parameter ε is calculated by formula (I.45). Having this in mind, and also considering equality (I.49), for the determination of limit load in the case of a macroscopic crack we obtain the following equality:

$$\lim_{s\to 0}\{\sqrt{s}\,\sigma_y(s,q_*)\}=\frac{2\sigma_0}{\pi}\sqrt{l_0\varepsilon_*}. \qquad (I.50)$$

Using equation (I.4) and (I.42), the right side of equality (I.50) is brought to the following to form:

$$\frac{2\sigma_0}{\pi}\sqrt{l_0\varepsilon_*}=\frac{K}{\pi}, \qquad (I.51)$$

where for plane deformation

$$K = \sqrt{\frac{\pi E \gamma}{1 - \nu^2}};$$

for the plane generalized state of strain

$$K = \sqrt{\pi E \gamma}.$$

Subsequently for determination of the limit load for a brittle body weakened by a macroscopic crack, we will use basically equality (I.47) or equality (I.50).

7. Maximum Equilibrium of an Unbounded Plane with a Linear Crack, When Concentrated Forces are Applied to its Sides

Let us examine an unbounded plate of unit thickness, weakened by a linear isolated crack of length $2l_0$ (Fig. 15). Let us introduce system of rectangular cartesian coordinates x0y, assuming that the middle plane of the plate coincides with coordinate plane x0y, and the crack is located along the x-axis on segment $a_0 \leq x \leq b_0$, where a_0, b_0 are the abscissas of the ends of the crack ($2l_0 = b_0 - a_0$). At the sides of the crack in points (ξ_1, +0) and (ξ_1, -0) are applied to concentrated forces P equal in the magnitude and opposite in direction. (If the thickness of the plate is h, and concentrated forces applied to the sides of the crack is P_1, then $P = P_1/h$.)

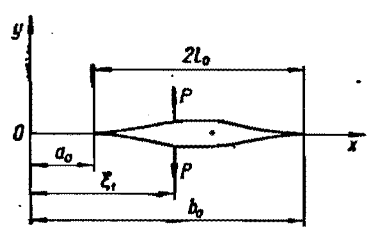


Fig. 15.

Let us determine the limit load for such a problem. For this we will use the formulated δ_k-model of a brittle body. In accordance

with the properties of this model the problem leads to the following problem of the mathematical theory of elasticity. In elastic plane xOy (Fig. 16) is a crack of length b-a where a, b are the abscissas of the boundary points between the elastically deformed part of the material and the crack. On the boundary of this crack in points $(\xi_1, +0)$ and $(\xi_1, -0)$ act normal tensile concentrated forces P and on sections $a \leq x < a_0$ and $b_0 < x \leq b$ – normal pressure $(-\sigma_0)$. It is necessary to determine vertical displacements v(x, 0) of the sides of the crack and parameters b and a.

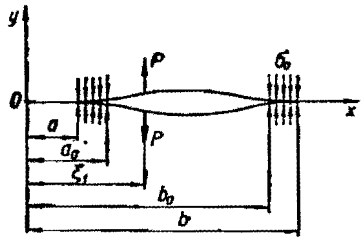


Fig. 16.

If we use expression (I.8) and transform[4] function $\Gamma(l, x, \xi)$ in reference to the system of coordinates shown on Fig. 16, then to determine vertical displacements v(x, 0) of the sides of the examined crack we obtain the following formula:

$$v(x, 0) = -cPH(a, b, x, \xi) + c\sigma_0 \left\{ \int_a^{a_0} H(a, b, x, \xi)\, d\xi + \int_{b_0}^{b} H(a, b, x, \xi)\, d\xi \right\}, \qquad (I.52)$$

where $a \leq x \leq b$; c – constant equal for generalized plane stress to $1/\pi E$; $H(a, b, x, \xi)$ – weighting function (Green's function) for the examined crack

$$H(a, b, x, \xi) = \ln \frac{(b-x)(x-a) + (\xi - x)\left(\frac{a+b}{2} - x\right) - \sqrt{(b-x)(x-a)(b-\xi)(\xi-a)}}{(b-x)(x-a) + (\xi - x)\left(\frac{a+b}{2} - x\right) + \sqrt{(b-x)(x-a)(b-\xi)(\xi-a)}}. \qquad (I.53)$$

To facilitate further calculations let us note that

$$\int H(a,b,x,\xi)\,d\xi = 2\sqrt{(b-x)(x-a)}\arccos\frac{2\xi-a-b}{b-a} - \\ -(x-\xi)H(a,b,x,\xi);$$

$$H'_x(a,b,x,\xi) = \frac{2\sqrt{(b-\xi)(\xi-a)}}{(x-\xi)\sqrt{(b-x)(x-a)}};$$

$$H'_\xi(a,b,x,\xi) = \frac{2\sqrt{(b-x)(x-a)}}{(\xi-x)\sqrt{(b-\xi)(\xi-a)}}. \tag{I.54}$$

Using formulas (I.54) and calculating integrals in equation (I.52), we find

$$v(x,0) = -cPH(a,b,x,\xi_1) + c\sigma_0\Big\{(x-b_0)H(a,b,x,b_0) - \\ -(x-a_0)H(a,b,x,a_0) + 2\sqrt{(b-x)(x-a)}\Big(\arccos\frac{2a_0-b-a}{b-a} - \\ -\pi - \arccos\frac{2b_0-b-a}{b-a}\Big)\Big\} \quad (a \leqslant x \leqslant b). \tag{I.55}$$

According to formula (I.30) on the contour of an opening crack the following condition must always hold:

$$\lim_{x\to a} v'_x(x,0) = 0; \quad \lim_{x\to b} v'_x(x,0) = 0. \tag{I.56}$$

Differentiating expression (I.55) with respect to x and satisfying equalities (I.56), we obtain

$$-\frac{2P\sqrt{(b-\xi_1)(\xi_1-a)}}{a-\xi_1} + \sigma_0\{(b-a)f_1(a,b,a_0,b_0) + 2f_2(a,b,a_0,b_0)\} = 0;$$

$$-\frac{2P\sqrt{(b-\xi_1)(\xi_1-a)}}{b-\xi_1} + \sigma_0\{-(b-a)f_1(a,b,a_0,b_0) + \\ + 2f_2(a,b,a_0,b_0)\} = 0, \tag{I.57}$$

where

$$f_1(a, b, a_0, b_0) = \arccos\frac{2a_0 - b - a}{b - a} - \pi - \arccos\frac{2b_0 - b - a}{b - a};$$
$$f_2(a, b, a_0, b_0) = \sqrt{(b - b_0)(b_0 - a)} - \sqrt{(b - a_0)(a_0 - a)}. \qquad (I.58)$$

Equalities (I.57) express the connection between load P and parameters b and a for any value $P \leq P_*$. Using these equalities and subtracting from the first equality the second, we find

$$P = -\sigma_0\sqrt{(b - \xi_1)(\xi_1 - a)}\, f_1(a, b, a_0, b_0). \qquad (I.59)$$

On the basis of expression (I.59) the first equation of system (I.57) can be written as:

$$2f_2(a, b, a_0, b_0) - (b + a - 2\xi_1) f_1(a, b, a_0, b_0) = 0. \qquad (I.60)$$

According to equality (I.59), formula (I.55) can be brought to the form

$$\begin{aligned} v(x, 0) = c\sigma_0 \{&\sqrt{(b - \xi_1)(\xi_1 - a)}\, f_1(a, b, a_0, b_0) H(a, b, x, \xi_1) + \\ &+ (x - b_0) H(a, b, x, b_0) - (x - a_0) H(a, b, x, a_0) + \\ &+ 2\sqrt{(b - x)(x - a)}\, f_1(a, b, a_0, b_0)\}. \end{aligned} \qquad (I.61)$$

Inasmuch as for the examined problem external load P is asymmetric with respect to the ends of the crack and $(\xi_1 - a) \leq (b - \xi_1)$, condition (I.5) for determination of limit load $P = P_*$ takes the form

$$v(a, 0) = \frac{1}{2}\delta_k.$$

Using formula (I.61), this condition is written as:

$$\sqrt{(b_1-\xi_1)(\xi_1-a_*)}\, f_1(a_*, b_1, a_0, b_0)\, H(a_*, b_1, a_0, \xi_1) - \\ -(b_0-a_0)\, H(a_*, b_1, a_0, b_0) + \\ +2\sqrt{(b_1-a_0)(a_0-a_*)}\, f_1(a_*, b_1, a_0, b_0) = \frac{\delta_k}{2c\sigma_0}, \tag{I.62}$$

where b_1 is the value of abscissa b, when parameter a reaches limit magnitude a_*.

This is one of the equations for determination of parameter b_1 and a_*. Another equation we obtain from (I.60), assuming that $b = b_1$ and $a = a_*$:

$$2f_2(a_*, b_1, a_0, b_0) = (b_1 + a_* - 2\xi_1)\, f_1(a_*, b_1, a_0, b_0), \tag{I.63}$$

where functions f_1 and f_2 are represented by equalities (I.58).

If parameters b_1 and a_* are determined, then according to equality (I.59) the value of maximum load $P = P_*$ for the examined problem (Fig. 16) is calculated by the formula

$$P_* = -\sigma_0 \sqrt{(b_1-\xi_1)(\xi_1-a_*)}\, f_1(a_*, b_1, a_0, b_0). \tag{I.64}$$

A special case of this problem, when forces P are applied across the middle of the crack, was investigated work [113]. The case of a macroscopic crack is investigated more specifically below (practically concentrated force can be applied only to the sides of such cracks). Let us introduce the following parameters:

$$\varepsilon_1 = \frac{b-b_0}{b_0-a_0} \quad \text{and} \quad \varepsilon_2 = \frac{a_0-a}{b_0-a_0}. \tag{I.65}$$

For a macroscopic crack parameters ε_1 and ε_2 are positive. They are small as compared to unity. From equalities (I.65) we find

$$b = b_0 + (b_0 - a_0)\,\varepsilon_1; \quad a = a_0 - (b_0 - a_0)\,\varepsilon_2;$$
$$b_1 = b_0 + (b_0 - a_0)\,\varepsilon_1^0; \quad a_* = a_0 - (b_0 - a_0)\,\varepsilon_2^*. \tag{I.66}$$

Expression (I.58) can be transformed to the form

$$f_1(a_1, b_*, a_0, b_0) = -\arccos\frac{1-\eta_1}{1+\eta_2} - \arccos\frac{1+\eta_1}{1+\eta_2};$$
$$f_2(a_1, b_*, a_0, b_0) = (b_0 - a_0)\left[\sqrt{(1+\varepsilon_2^*)\,\varepsilon_1^0} - \sqrt{(1+\varepsilon_1^0)\,\varepsilon_2^*}\right], \tag{I.67}$$

where $\eta_1 = \varepsilon_2^* - \varepsilon_1^0$; $\eta_2 = \varepsilon_1^0 + \varepsilon_2^*$; values of ε_1^0 and ε_2^* are determined by equalities (I.66).

Expressing in formulas (I.62)-(I.64) the quantities b_1 and a_* by the parameters correspondingly ε_1^0 and ε_2^*, expanding the obtained expressions in a series in small parameters ε_1^0 and ε_2^* and disregarding quantities of the order of $0\left(\varepsilon^{\frac{n+1}{n}}\right)$, where $n \geq 1$, we obtain

$$(b_0 - 2\xi_1 + a_0)\,\varepsilon_2^* + 2(b_0 - \xi_1)\sqrt{\varepsilon_1^0\varepsilon_2^*} = \frac{\delta_k}{8c\sigma_0}; \tag{I.68}$$

$$(b_0 - \xi_1)\sqrt{\varepsilon_1^0} = (\xi_1 - a_0)\sqrt{\varepsilon_2^*}.$$
$$P_* = 2\sigma_0\sqrt{(b_0 - \xi_1)(\xi_1 - a_0)}\,(\sqrt{\varepsilon_1^0} + \sqrt{\varepsilon_2^*}). \tag{I.69}$$

The solution of system (I.68) has the form

$$\left.\begin{aligned} &2l_0\varepsilon_2^* = a_0 - a_* = \frac{\delta_k}{8c\sigma_0};\\ &\varepsilon_1^0 = \frac{b - b_0}{2l_0} = \frac{(\xi_1 - a_0)^2}{(b_0 - \xi_1)^2}\,\varepsilon_2^*;\\ &2l_0 = b_0 - a_0,\ \xi_1 - a_0 \ll b_0 - \xi_1. \end{aligned}\right\} \tag{I.70}$$

Expression (I.70) for ε_2^* is similar to formula (I.46) and shows the autonomy of the dead-end part of a maximum equilibrium macroscopic crack for the examined problem.

According to equalities (I.69) and (I.70)

$$P_* = \frac{P_{1*}}{h} = K\sqrt{\frac{(b_0 - a_0)(\xi_1 - a_0)}{b_0 - \xi_1}}, \qquad (I.71)$$

where h is the thickness of the plate.

For practical calculations of the limit load $P = P_*$ in the case of a macroscopic crack of length $2l_0 = b_0 - a_0$ (see Fig. 15), when concentrated forces P are applied to the sides of a crack asymmetrically relative to its ends, formula (I.71) should be written as:

$$P_* = K\sqrt{\frac{2l_0\xi_0}{(2l_0 - \xi_0)}} \quad \left(P_* = \frac{P_{1*}}{h}\right), \qquad (I.72)$$

where ε_0 is the minimum distance between points of application of forces P and ends of crack $\xi_0 = \xi_1 - a_0$.

Formula (I.72) is easily obtained also directly from equation (I.47). Actually, by transforming expression (I.9) to the system of coordinates shown on Fig. 15, we obtain

$$\sigma_y^0(x, 0) = \frac{P\sqrt{(b_0 - \xi_1)(\xi_1 - a_0)}}{\pi\sqrt{(b_0 - x)(a_0 - x)}\,|x - \xi_1|}, \qquad (I.73)$$

where $a_0 \leq \xi_1 \leq b_0$; $x < a_0$, or $x > b_0$.

Putting expression (I.73) into equality (I.47) and noticing that when $x \leq a$ we have $s = (a_0 - x)$, after simple transformations

we obtain formula (I.71). Assuming that $a_0 = -b_0$ and $\xi_1 = 0$, we obtain the formula for determination of load P_{1*}, when concentrated forces are applied symmetrically with respect to the ends of the crack:

$$P_{1*} = hK\sqrt{2l_0}, \tag{I.74}$$

where $l_0 = b_0$ is the half-length of the crack. (Formula (I.74) was obtained earlier in works [5, 113.])

From formula (I.74) it follows that the greater the length of a rectilinear crack in an unbounded plate, the greater the concentrated forces $P_1 = P_{1*}$ necessary for its propagation. This means that in this case there is a stable propagation of the crack (for comparison see p. xvii).

From equality (I.71) it is easy to obtain a formula for determination of limit load $P_1 = P_{1*}$, when an infinite plate is weakened by a linear semi-infinite crack (Fig. 17), and at a distance ξ_0 from the end of the crack concentrated forces P_1 are applied. For this it is necessary to consider that $\xi_1 - a_0 = \xi_0$ and $b_0 \to \infty$. Carrying out also transformation in formula (I.71), we obtain

$$P_{1*} = hK\sqrt{\xi_0}. \tag{I.75}$$

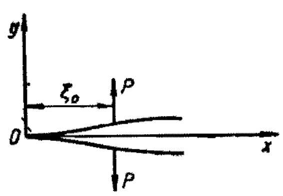


Fig. 17.

For a plate with a semi-infinite cut, assuming that $b_0 \to \infty$, $a_0 \to 0$, $\xi_1 = \xi_0$, from formula (I.73) we find

$$\sigma_y^0(x, 0) = \frac{P\sqrt{\xi_0}}{\pi\sqrt{-x}\,|x-\xi_0|} \text{ when } x < 0. \tag{I.76}$$

Furthermore, using formula (I.52), it is possible to construct also weighing function $H_0(x, \xi)$. For this in equation (I.52) we can take $a = 0$, $b \to \infty$ and carry out the necessary transformations. Then

$$H(0, \infty, x, \xi) = H_0(x, \xi) = 2 \ln \frac{|\sqrt{\xi} - \sqrt{x}|}{\sqrt{x} + \sqrt{\xi}}. \tag{I.77}$$

With this function one can determine the shape of an opening crack in the form of a semi-infinite cut in a plate (see Fig. 17), when arbitrary normal loads are applied to the sides of the crack, vanishing in points at infinity. Function (I.77) was obtained earlier by other means in work [64].

Footnotes

[1]The representation of a crack in the form of a cut in a material continuum (elastic model) can be considered as justified, inasmuch as the characteristic linear dimension of a real crack is always considerably larger than the maximum distance between its edges.

[2]Such an approach to the state of strain in a body with cracks essentially means therrepresentation of the state of strain in the form of the sum: 1) of the state of strain in a body without a crack under a preassigned system of external stresses; 2) of the stressed state in a such body with cracks to whose sides are applied normal pressures equal in magnitude to the stresses appearing in a solid in the region of the cracks.

Inasmuch as the first state of strain depends on the dimensions of the crack, the conditions of crack propagation are wholly determined on the basis of a second state of strain.

[3]These properties of the structure of the edge of an equilibrium crack are established on the basis of other considerations for the case of rather well-developed cracks also in works [5 and 8].

[4]This requires replacing variables x and ξ in formula (I.8) by the variables correspondingly $x - l - a$ and $\xi - l - a$ and to set $l = \frac{b-a}{2}$.

CHAPTER II

METHODS OF DETERMINING EFFECTIVE SURFACE ENERGY

1. Introductory Remarks

To determine the limit values of an external load for a solid with cracks it is necessary to know the effective surface energy γ of the given material. This energy in numerical expression constitutes the work which it is necessary to expend on the formation of a unit surface of a given material under preassigned conditions (temperature, pressure, environment). This characteristic of a material is connected with modulus of cohesion K by formulas (22).

In the case of an ideally brittle break, when the new surface forms without microplastic deformations, effective surface energy is an averaged value of the true surface energy of the material. If, however, the formation of the surface of a body is accompanied by microplastic deformations in the subsurface layer of a new surface, i.e., when the formation of the new surface is quasi-brittle, the value of the effective surface energy will exceed the true value of the specific surface energy of the given material. In this case the value of γ is the sum of the specific surface energy of a material γ_0 and the density of energy of microplastic deformations γ_Π appearing in the subsurface layer of the forming surface.

Direct measurement of γ involves serious difficulties, therefore there are still no effective methods of measurement. A survey of research in given area is contained in works [34, 62, 187]. Below the

possibility of using results of the theory of equilibrium fractures is discussed.

2. Experimental Determination of Effective Surface Energy of Brittle Materials

For direct measurement of γ selection of the loading pattern of a plate with a crack is very important. It should be such that propagation of a crack in the body (formation of new surfaces) is **stable** and not spontaneous. Under such a condition in any moment of the deformation of a body crack propagation can be stopped by stopping the increase of the external load; the body can be unloaded and the work expended on the formation of a new surface can be determined.

Having this in mind, effective surface energy can be determined in the following way [129]. From a brittle material samples are prepared – plates a × 2L in size (Fig. 18). In the center of every plate a small hole is drilled. Along the diameter of hole a macroscopic crack of length $2l_0$ is made. The plate with the crack is loaded using a special device (Fig. 19) by forces P in accordance with the pattern shown on Fig. 18. With this load pattern the macroscopic crack in the plate at first opens when forces P increases, reaches the maximum equilibrium state, and then starts to spread over the cross section of the plate (dotted line on Fig. 18). In our case of loading the plate crack propagation will be stable if $l_0 \ll L$ (see formula (I.74)).

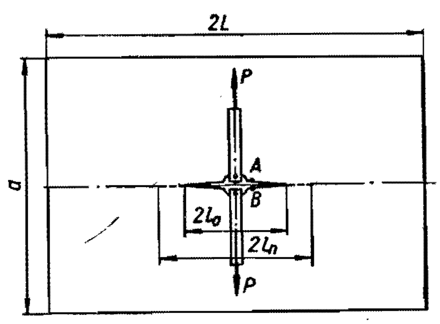


Fig. 18.

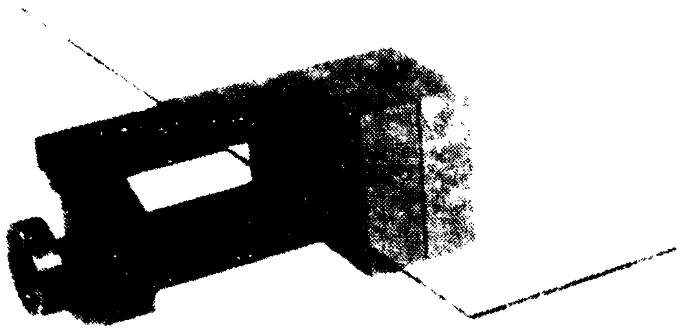


Fig. 19.

Let us examine two points B and A (see Fig. 18), located on the opposite sides of the crack in the neighborhood of the hole. Let us assume that v is the displacement of these points relative to one another during the deformation of the plate by forces P. If for every value of P, v can be measured, then in plane (P, v) it is possible to construct curve $\vec{P}_n$ (Fig. 20). The area bounded by this curve, axis Ov and vertical line $v = v_n$ is equal to the work done by forces P as they increase from zero to the value of P_n.

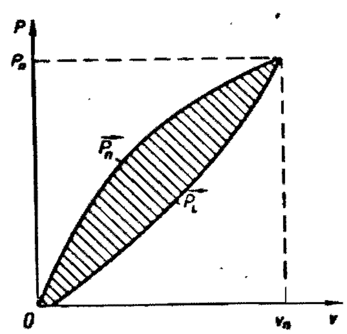


Fig. 20.

Let us designate this work through A_1. Let us assume that as a result of the increase of forces P from 0 to P_n the crack length increased from $2l_0$ to $2l_n$. Unloading the plate, i.e., decreasing load P, we measure v for $P_i < P_n$. According to obtained data in plane (P, v) we construct curve $\overleftarrow{P}_i$ (Fig. 20). The area bounded by curve $\overleftarrow{P}_i$, axis Ov and line $v = v_n$ is equal to work A_2, which was

liberated by the system during unloading.

Thus, when a plate with a crack is loaded by forces P_n (see Fig. 18), as a result of which the crack propagates by $2\Delta l = 2(l_n - l_0)$, and there is subsequent unloading of the system we have following balance of work:

$$\Delta A = A_1 - A_2. \tag{II.1}$$

The magnitude irreversible work ΔA in this case constitutes in numerical expression the area bounded by curves $\vec{P}_n$ and $\overleftarrow{P}_i$ (shaded region on Fig. 20). As a result of the deformation of the plate with the crack (loading - unloading by forces P) in the plate will be formed new surfaces of general area

$$\Delta S = 2h2(l_n - l_0), \tag{II.2}$$

where h is the thickness of the plate.

If γ is the density of the effective surface energy of the given material (effective surface energy per unit surface), then the work expended on the formation in this material of a surface of area ΔS is determined by the equality

$$\Delta U = \gamma \Delta S = 4\gamma h(l_n - l_0). \tag{II.3}$$

Inasmuch as for the accepted loading pattern during a smooth increase of load P there occurs a smooth (without dynamic perturbation opening and propagation of a crack, then

$$\Delta A = \Delta U. \tag{II.4}$$

If load $P = P_n$ is small and crack propagation does not occur $(l_0 = l_n)$, then, according to (II.3), $\Delta U = 0$. In this case, as experiment shows, irreversible work ΔA also is equal to zero.

On the basis of equalities (II.3) and (II.4) we obtain the following formula:

$$\gamma = \frac{\Delta A}{4h\Delta l}. \qquad \text{(II.5)}$$

Thus, for determination of the density of effective surface energy of brittle materials it is necessary to stretch the plate (sample) according to the loading pattern shown on Fig. 18, to determine ΔA, to measure crack increase $\Delta l = l_n - l_0$, and then by formula (II.5) to calculate γ. Formula (II.5) can be used also to determine γ by another load pattern for the plate, for example by that shown on Fig. 21, but in this case

$$\Delta l = \frac{1}{2}(l_n - l_0).$$

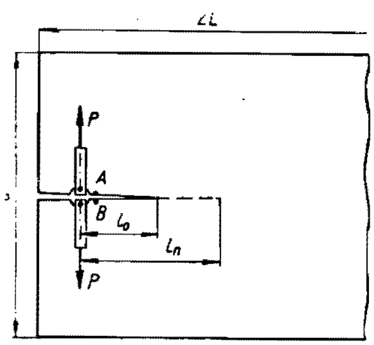


Fig. 21.

Consequently, the method of direct measurement of effective surface energy of brittle materials is based on measurement of the irreversible work which accompanies the propagation of a linear crack in a brittle plate, when the plate is loaded by monotonically increasing concentrated forces applied to the sides of the crack, and then is unloaded as a result of a monotonic decrease of these forces. This method of loading ensures stable crack propagation over the cross section of the plate and can directly measure the work expended on formation of new surfaces of a given material, i.e., directly measure γ.

Measurement of γ for silicate glass. The above method was used in work [129] for determination of the effective surface energy of silicate glass. To do this a glass sheet in the delivered state was made into plate samples (see Fig. 18) with the dimensions: $a = 40$ *см*; $L = 25$ *см*.

The glass had the following chemical composition: 72.68% SiO_2, 1.17% R_2O_3, 7.73% CaO, 3.8% MgO, 13.73% Na_2O, 0.89% SO_3.

In the center of every plate a hole approximately 1 cm in diameter was drilled, after which over the diameter of the hole a glass cutter was used to make a crack (cut). Then the plate was loaded according to Fig. 18 on a special rigid attachment (see Fig. 19), which opened the crack all over the thickness of the plate along the cut $2l_0$. The plate was loaded on a special installation (Fig. 22), where forces P on the sample and corresponding to distance v were measured.

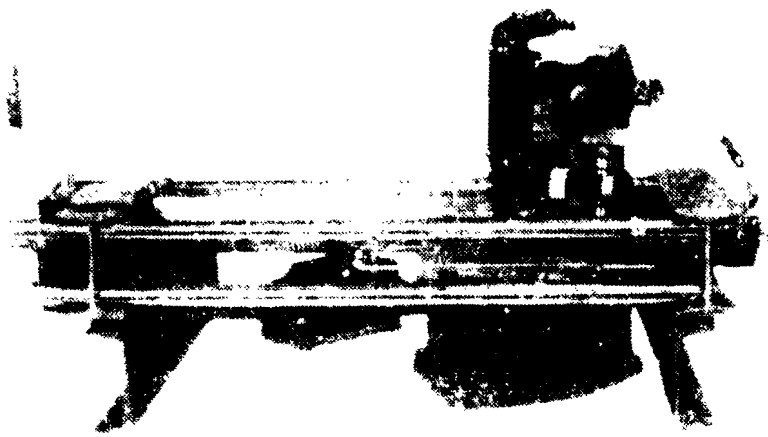
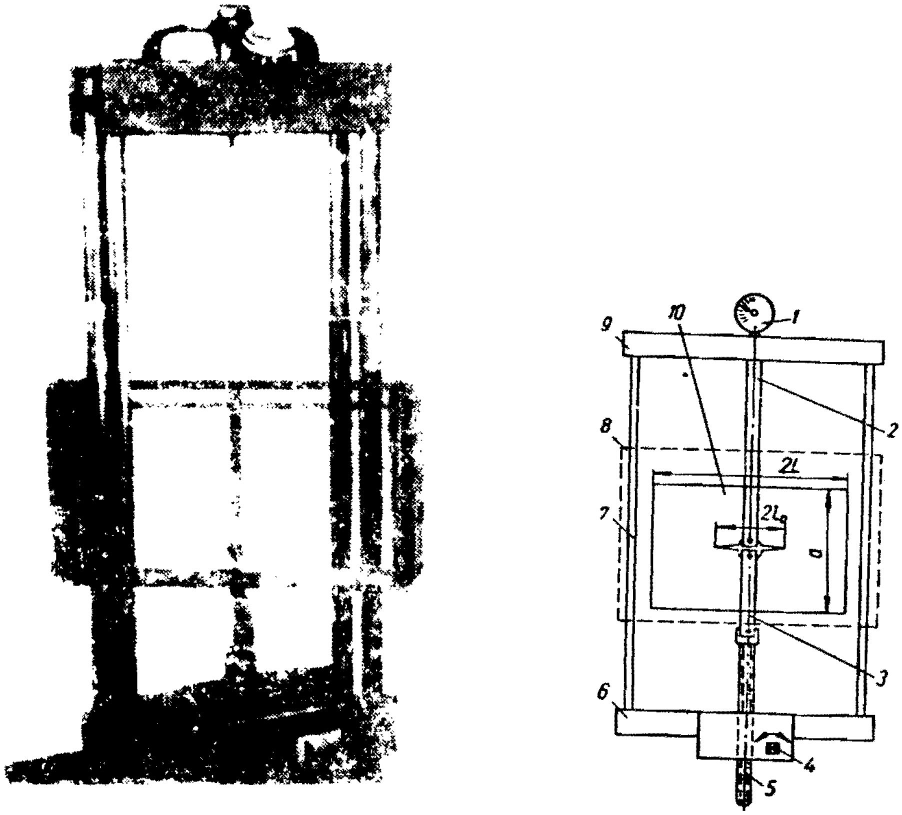


Fig. 22.

For this purpose the installation (Figs. 22 and 23), shown in Fig. 23b is used. Two I-beams 6 and 9 are interconnected by four supports 7. On beam 6 is placed a worm pair. When the handle of worm is turned to one side, the worm wheel — nut imparts smooth forward motion to screw 5. On beam 9 is dynamometer 1, which helps fix the value of forces P. Plate 10 is loaded through tension rods 3 and 2, joined with the plate through the hole. Roll 3 is connected also with screw 5, and rod 2 with dynamometer 1. Between the supports of the installation a microscope (type PMT-3) is built in, which

measures the opening of the crack, i.e., distance v between points A and B. Dish 8 retains the liquid medium, if it is required, for example, to study its effect on γ.



a b

Fig. 23.

On this installation the glass plate samples are loaded at room temperature until initial crack length $2l_0$, increasing, reaches a certain value $2l_n$; then distance v is measured for fixed values $P < P_n$ and curve $\vec{P}_n$ is built by these data. Then the load increase is stopped and the final crack length is measured. After that plate is unloaded and for fixed values $P < P_n$ again v is measured and the curve of $\overleftarrow{P}_i$ constructed. According to the measurements for every plate a diagram of irreversible work is constructed. Then with a planimeter the area of the diagram is calculated, i.e., ΔA. According to obtained data $(\Delta A, \Delta l_n)$ and by formula (II.5) the value of the effective surface energy for the material of every plate is calculated (Table 4). On Fig. 24 is the graph of irreversible work for one

plate of this series when $\Delta l = 2{,}6$ см; $\Delta A = 2{,}3 \cdot 10^{-4}$ J; $\gamma = 2{,}01 \cdot 10^{-4}$ J/cm^2 (shaded circles on the figure correspond to loaded nonshaded to unloaded).

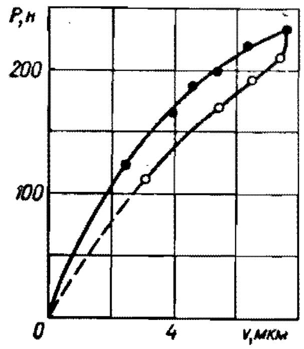


Fig. 24.

On Fig. 25 is the curve of the loading-unloading change of a plate with a crack, when load P does not cause crack propagation ($P < P_*$), i.e., when $\Delta l_n = 0$. In this case, as can be seen from Fig. 25, within the limits of experimental accuracy $\Delta A = 0$. The mean value of the effective surface energy of silicate glass $\gamma_{cp} = 2{,}1 \cdot 10^{-4}$ J/cm^2 (Table 4).

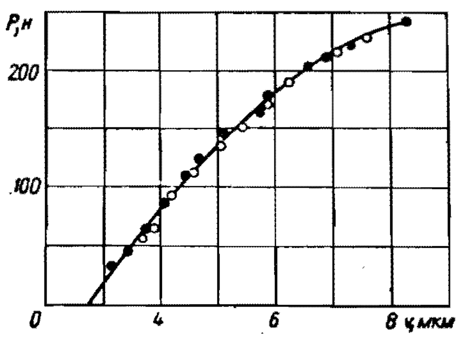


Fig. 25.

Table 4.

No. plates	h, см	$2l_0$, см	$2l_n$, см	$2\Delta l$, см	$\Delta A \cdot 10^4$, J	$\gamma \cdot 10^4$, J/cm^2
1	0,26	8,0	9,9	1,9	2,2	2,2
2	0,25	4,8	7,6	2,8	3,2	2,3
3	0,25	7,6	14,2	6,6	6,6	2,0
4	0,22	13,6	16,3	2,7	2,4	2,0
5	0,22	16,3	18,8	2,5	2,0	1,8
6	0,22	18,8	21,2	2,4	2,6	2,4
7	0,22	21,2	23,8	2,6	2,3	2,0
8	0,22	23,8	28,1	4,8	4,2	2,2

For silicate glass γ was likewise measured according to the loading pattern shown on Fig. 21. With this pattern analogous diagrams of irreversible work ΔA are obtained (Fig. 26). Results of measurements are given in Table 5 ($\gamma_{cp} = 2{,}2 \cdot 10^{-4}$ J/cm^2).

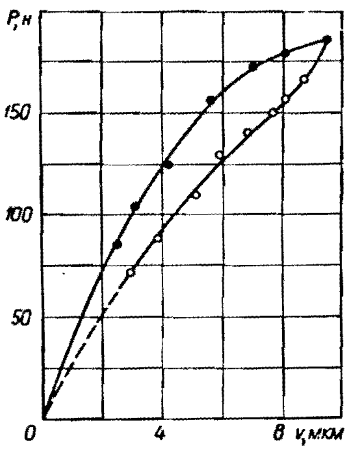


Fig. 26.

Table 5.

No. plates	h, cм	l_0, cм	l_n, cм	$2\Delta l$, cм	$\Delta A \cdot 10^4$, J	$\gamma \cdot 10^4$, J/cm^2
1	0,28	4,3	5,5	1,2	1,3	1,9
2	2,28	6,2	7,8	1,3	2,2	2,2
3	0,28	5,2	6,4	1,2	1,7	2,6
4	0,28	6,4	7,2	0,8	1,1	2,4
5	0,28	4,2	5,2	1,0	0,9	1,6
6	0,53	5,9	6,9	1,0	2,3	2,2
7	0,53	6,9	7,7	0,8	1,9	2,2
8	0,53	7,7	9,0	1,3	3,1	2,2

Determination of γ for plastic. The above method (just as it was done for silicate glass) was used to measure of γ for plastic (plexiglas). Tests were conducted on plates 23 × 47 cm (Table 6). According to obtained data for plastic $\gamma_{cp} = 1{,}5 \cdot 10^{-2}$ J/cm^2.

If we substitute this value of γ_{cp} into the formula for determination of modulus of cohesion $K_{cp} = \sqrt{\pi E \gamma_{cp}}$ and consider that for plastic $E = 2{,}45 \cdot 10^5$ N/cm^2, then we obtain $K_{cp} \approx 1100$ N/cm$^{3/2}$. This value is close to the value $K = 1500$ N/cm$^{3/2}$ given in work [5].

Table 6.

No. plates	h, см	$2l_0$, см	$2l_n$, см	$2\Delta l$, см	$\Delta A \cdot 10^1$, J	$\gamma \cdot 10^1$, J/cm²
1	0,58	2,5	4,4	1,9	3,6	1,6
2	0,58	4,4	6,7	2,3	3,4	1,3
3	0,58	6,7	8,8	2,1	3,9	1,6
4	0,58	8,8	14,3	5,5	8,2	1,3
5	0,59	3,0	5,5	2,5	4,3	1,4
6	0,59	5,5	8,5	3,0	5,5	1,6
7	0,59	8,5	11,3	2,8	5,4	1,6
8	0,59	11,3	14,6	3,3	6,4	1,6

3. Determination of γ on the Basis of Formulas of the Theory of Crack Propagation

In Section 7, Chapter I, it is shown that in the case of extension of an infinite plate with a linear macrocrack of length $2l_i$, when concentrated forces P are applied to the sides of the crack an equal distance from its ends, the connection between the length of the crack and limit load P_* is expressed by the formula

$$P_{*i} = Kh\sqrt{2l_i} \quad (i = 0, 1, 2, 3\ldots),$$

where h is the thickness of the plate.

From this formula we find

$$K = \frac{P_{*i}}{h\sqrt{2l_i}};$$

$$\gamma = \frac{P_{*i}^2}{2h^2\pi E_1 l_i}. \qquad \text{(II.6)}$$

Here for generalized plane stress $E_1 = E$; for plane deformation $E_1 = \frac{E}{1-\nu^2}$, where E is Young's modulus; ν is Poisson's ratio.

If initial length $2l_i$ of the cracks is not too great and in a bounded plate the stressed-deformed state induced by forces P in the environment of the crack can be considered as coinciding with such in an infinite plate with a crack of the same length and with the same loads, then formula (II.6) can be used for determination of the effective surface energy of the material. For this it is necessary to determine the value of P_{*i} for fixed crack length $2l_i$ and then to calculate γ by the formula (II.6).

It is difficult to apply concentrated forces directly to the sides of a linear crack. However, this difficulty is easily removed if in the plate a small hole is drilled (see Fig. 18) and the plate is examined as weakened by a circular hole with radial cracks. Such a problem differs from ours, but, as calculations show [8, 117] (see Chapter V), the dependence of load P_* on length $2l_i$ for these problems practically coincides if $\frac{l_i}{R} > 1{,}5$, where R is the radius of the circular hole. Having this in mind, in works [127, 129] formula (II.6) was used to determine γ for silicate glass. The obtained mean values of γ are compared with data of Section 1 of this chapter.

Determination of γ for silicate glass. For the experiments a batch of glass plates 25 × 25 cm was made (chemical composition of glass is shown on page 40). In the center of every plate a hole approximately 1 cm in diameter was drilled. Over the diameter of the hole in the plate was formed a through linear crack 4-5 cm long. Such plates were loaded (see Fig. 18) on a special installation (see Fig. 23). When the plate was loaded by forces P for fixed values $2l_i > 2l_0$ maximum force P_{*i} was measured (i = 1, 2, 3...) as long as the influence of the edge of the plate had no effect.

Results of experimental measurements of P_{i*} (averaged over each plate) are given in Table 7.

Table 7.

No. plates	$2l_i$, cm	h, cm	$P_{*i} \cdot 10^{-1}$, N	$K \cdot 10^{-1}$, N/cm$^{3/2}$	$\gamma \cdot 10^4$, J/cm^2
1	5,0	0,22	34	69	2,1
2	6,0	0,22	37	69	2,1
3	6,6	0,22	38	67	2,0
4	5,7	0,28	44	66	2,0
5	6,0	0,28	47	68	2,1
6	5,5	0,28	43	66	2,0
7	4,7	0,28	39	64	1,8
8	4,9	0,28	41	66	2,0
9	5,7	0,28	46	69	2,1
10	5,5	0,27	47	74	2,5
11	6,0	0,27	48	72	2,3
12	7,4	0,27	53	72	2,3
13	3,9	0,40	58	73	2,4
14	5,8	0,40	68	70	2,2
15	8,4	0,40	76	66	2,0
16	5,3	0,56	88	68	2,1
17	6,3	0,56	98	70	2,2
18	7,2	0,56	108	72	2,3
19	4,6	0,58	83	67	2,0
20	6,1	0,58	98	68	2,1
21	7,7	0,58	106	66	2,0

Using these data and formula (II.6), it is possible to determine easily the density of effective surface energy. Values of γ calculated by this (direct) method for silicate glass (Table 7, $\gamma_{cp} = 2{,}1 \cdot 10^{-4}$ J/cm^2; $E = 6{,}7 \cdot 10^6$ N/cm^2; $\nu = 0.24$) will agree well with results of direct measurement of this characteristic (see Table 5).

Determination of γ for carbon steel [59]. The value of modulus of cohesion K and effective surface energy γ for carbon steel U8 were determined on samples of sheet steel 36 × 18 cm, made from plates 2-2.5 mm thick. In the center of every plate sample an initial concentrator cut of a definite length was created, and a hole was cut to connect rods when the plate was loaded. To ensure quasi-brittle rupture the samples with the cut were heated to 810 ± 10°C in an electric furnace, and then cooled (quenched) in oil at 20-25°C. The samples were then tempered: heated to 190°C and cooled in the furnace. After that, to remove the scales and waviness, the plate samples were ground from both sides to a thickness of 1.5 mm. The grinding was done on a flat grinder with minimum vertical supply and abundant cooling to avoid cold hardening and heating of the subsurface layer. To checking the homogeneity of the samples after the heat treatment their Rockwell hardness was determined and was within the limits of 62-66 HRC.

The prepared plate samples were loaded by forces P according to Fig. 18.

To obtain an initial cut of natural sharpness in the dead-end part of a crack, before beginning the experiment every plate is loaded on a rupture-test machine without a dynamometer (see Fig. 23) to a load P at which the length of the cut increased all over the thickness of the plate by 6-10 mm. After that the plate is loaded as is shown in Fig. 23 (rods connected to dynamometer) and P_{*i} is measured for every crack with fixed length $2l_i$ (points on Fig. 27).

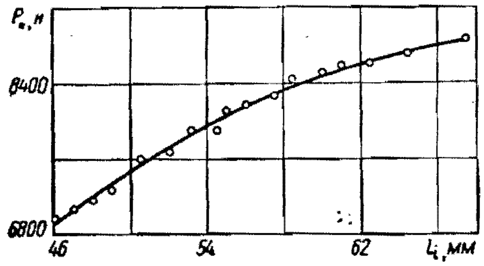


Fig. 27.

According to measurements and by formula (II.6) the mean value of the modulus of cohesion is calculated: $K_{cp} = 2{,}27 \cdot 10^4$ N/cm$^{3/2}$. At a given value of K_{cp} the curve calculated by formula (I.74), will fall on the experimental points of Fig. 27.

On the basis of these data ($K_{cp} = 2{,}27 \cdot 10^4$ N/cm$^{3/2}$ and $E = 2{,}06 \cdot 10^7$ N/cm^2) for quenched carbon steel U8 we have $\gamma_{cp} = 7{,}5 \cdot 10^{-2}$ J/cm^2.

The obtained value of γ is many times larger than the assumed value of the true surface energy of metals (approximately 10^{-4} J/cm^2). This is because in the surface layer of metals the formation of a new surface is accompanied by microplastic deformations, on which considerable energy is expended. In work [186] it is noted that for low-carbon steel the thin layer adjacent to the rupture surface is plastically deformed even when macroscopic rupture can be considered practically brittle.

According to X-ray analysis conducted by Fehlbeck and Orowan [186], for low-carbon annealed steel the specific work of plastic deformation is $\gamma_n = 2 \cdot 10^{-1}$ J/cm^2. Consequently, in quasi-brittle rupture the most work in the formation of a unit of new surface will be the work of plastic deformation in the rupture area.

4. Use of the Fracture Method for Determining γ on Small Samples

Above a method of determining the effective surface energy of brittle materials, when from the given materials it was possible to prepare sufficiently large plate samples is described. In certain cases (for example, determination of γ for single crystals) samples are comparatively small, and directly analytic solutions obtained for an infinite plate with a crack, cannot be used, yet it is difficult to obtain plates of large single crystals. In connection with this for the experimental determination of γ it is expedient to use analytic relationships, characterizing the propagation of cracks in plates (strips) of finite width.

Propagation of a linear fracture in a strip. Let us assume that a strip of finite width 2L and thickness h (Fig. 28), weakened by a linear through isolated crack of length $2l_0$, is stretched by concentrated forces P, applied to the sides of the crack an equal distance from the ends. Let us determine limit load $P = P_*$.

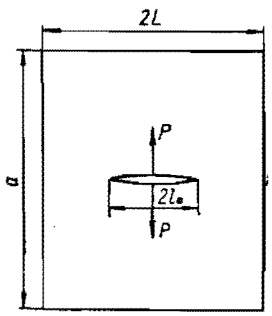


Fig. 28.

The exact resolution of the problem involves great mathematical difficulties. Therefore below we will give an approximate solution obtained in works [5, 12] on the basis of the method of successive approximations, developed in works [95 and 172]. As the first approximation we take the field of stresses in a strip $(-L \leqslant x \leqslant L;$ $-\infty < y < \infty)$ of an infinite plate with a periodic system of cracks of length $2l_0$, located along the line (axis Ox) with centers at points $x = \pm 2nl$, where n is an integer. It is assumed that to the sides of every crack are applied external loads just as for the examined strip (see Fig. 28). In this case on lines x = ±L tangential stresses are equal to zero, and the normal stresses are different from zero. As was noted in work [8], normal stresses, coinciding with the direction of the plane of the crack, do not essentially influence the propagation of a linear crack. Because of this we can limit ourselves to only the first approximation.

Elastic rupture stresses $\sigma_y(x, 0)$ for the examined strip (Fig. 28) in the first approximation are easily determined according to the solution obtained in work [197]:

$$\sigma_y(x, 0) = \frac{P \sin \frac{\pi l_0}{2L}}{2hL \sin \frac{\pi x}{2L}} \left[\sin^2 \frac{\pi x}{2L} - \sin^2 \frac{\pi l_0}{2L}\right]^{-1/2}. \qquad \text{(II.7)}$$

Near the ends of the crack, i.e., at $x = \pm(l_0 + s)$, where s is small, we have

$$\sigma_y(s, 0) = \frac{1}{h\sqrt{2\pi L \sin\frac{\pi l_0}{L}}} \cdot \frac{P}{\sqrt{s}} + 0(1). \tag{II.7a}$$

Then on the basis of equality (I.47) in our case

$$P_* = K\frac{h}{\pi}\sqrt{2\pi L \sin\frac{\pi l_*}{L}}. \tag{II.8}$$

For the plane generalized stressed state (thin plate) $K = \sqrt{\pi E \gamma}$ and, consequently, formula (II.8) can be written as:

$$\gamma = \frac{P_*^2}{2Eh^2 L \sin\frac{\pi l_0}{L}}. \tag{II.9}$$

Using approximate formula (II.9), it is easy to determine γ for comparatively small samples.

Note. If the transverse dimension 2L of the strip (see Fig. 28) is considerably more than $2l_0$, i.e., $l_0 \ll L$, then (accurate to small $\frac{l_0}{L}$ inclusively) from formulas (II.8), (II.9) follow formulas (II.6).

It was interesting to compare γ calculated according to experimental data by formulas (II.6) and (II.9). A series of plates from silicate glass with differing $\frac{l_l}{L}$ was prepared and tested (by the method described in Sections 1 and 2) at room temperature. Then by formulas (II.6), (II.9) for every plates values of γ were calculated [Fig. 29, where curve 1 is built according to the results of formula (II.9), and curve 2 according to the results of formula (II.6)]. According to Fig. 29, formula (II.6) can be used for determination of γ only when $\frac{l_l}{L} < 0{,}25$. Approximate formula (II.9) can be used for determination of γ of brittle materials on plates with any relationship $\frac{l_l}{L}$. However, for exact determination of γ on small samples it is necessary to have a strict resolution of problem for the

propagation of macrocracks in a plate of bounded dimensions (see, for example, works [24, 25, 79, 80, 173]).

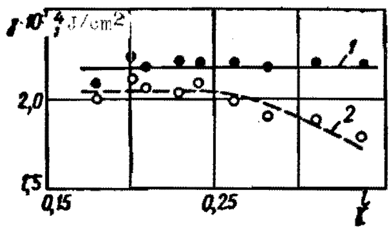


Fig. 29.

Determination of γ on samples of single NaCl crystals. The samples of these single crystals for experiment[1] were prepared using a setup which operated according to the Kyropoulos principle [3, 61]. The source material for growing the single crystals was chemically pure salt, containing ∿1% impurities. When the single NaCl crystals were grown, a large portion of the impurities remained in the remainder of melt in the crucible. Double crystallization achieved purer single crystals. At first several crystals were grown at an increased growth, then, observing optimum conditions of growth, from the obtained small crystals one crystal was grown for the experiments. Crystals grown in such a way were annealed at 740-750 $^{\circ}$C for 12-16 h, and then slowly cooled at a predetermined rate of 50 deg/h to a furnace temperature of 200 $^{\circ}$C, after which the furnace cooled further independently.

In such a way single NaCl crystals were made in the form of cylinders with an average diameter of approximately 50 mm and height of 80 mm. The cylindrical crystal was split into disks from 1-3 mm thick. Sharp knife from disks pricked rectangular plates (Table 8). In the center of every plate a water-moistened drill made a small hole (diameter 2 mm). Across the diameter of the hole the blade of a safety razor made a small crack of length $2l_0$ in the plate. So that during the formation of the initial crack the crystal plate did not split in two, around the edges of the plate (perpendicular to the direction of the crack) a thread was gently wound and removed after the crack was on. The dimensions of every sample and crack were measured under microscope accurate to 0.1 mm.

Table 8.

No. plate	h, см	$2l_0$, см	$2L$, см	P_*, N	$\gamma \cdot 10^7$, J/cm^2
1	0,28	0,65	1,70	53,4	470
2	0,26	0,50	1,30	32,6	270
3	0,28	0,70	1,40	48,3	440
4	0,26	0,65	1,40	42,6	400
5	0,30	0,70	1,40	45,0	330
6	0,29	0,75	1,80	43,3	260
7	0,24	0,55	1,38	37,0	370
8	0,27	0,65	1,82	45,1	350
9	0,30	0,75	1,15	41,4	380
10	0,20	0,68	1,17	27,5	340
11	0,26	0,64	1,27	30,4	220
12	0,20	0,62	1,38	34,8	460

The prepared samples of single NaCl crystals were loaded by forces P according to Fig. 30, and for every crack length $(2l_0)$ limit load P was measured (accurate to 25 g). All measurements were conducted at room temperature. According to the data of these measurements for determination of effective surface energy γ (or K) formula was used (II.9) (for NaCl we have $E = 4{,}87 \cdot 10^6$ N/cm^2).

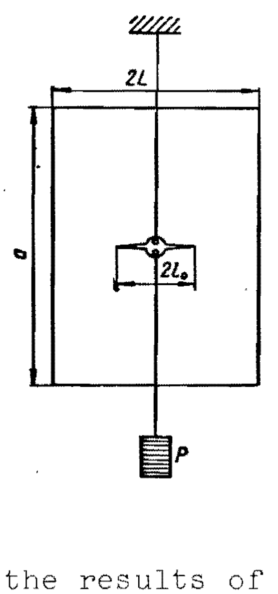


Fig. 30.

According to the results of the experiments it has been established that for NaCl $\gamma_{cp} = 360 \cdot 10^{-7}$ J/cm^2 (Table 8). According to Table 9 it may be concluded that values of γ for NaCl obtained on the basis of the method of cracks will agree with experimental data of works [62, 174, 193, 205]. Deviation of experimental and theoretical data,

is apparently explained by the fact that theoretical values of the surface energy of ionic crystals of NaCl type are calculated for crystals of ideal structure at a temperature of absolute zero. The single crystals of NaCl used in experiments did not have an ideal, completed structure. The γ was measured under room conditions (temperature and humidity), which could condition an imperfect brittle rupture, and, consequently, a certain expenditure of energy on the formation of microplastic deformations in the subsurface layer of the propagating crack.

Table 9.

Theoretical value $\gamma \cdot 10^7$, J/cm^2	Source	Experimental value $\gamma \cdot 10^7$, J/cm^2	Source
150	[176]	276	[174]
164	[162]	366	[193]
130	[35]	381	[205]
187	[229]	300	[62]
155	[218]	360	According to Table 8
172	[50]		
213	[51]		

In connection with this it is of interest to study properties of the subsurface layer of a developed crack and determine γ for a change (increase or lowering) of temperature of the body, and when the body is affected by the surrounding surface active medium.

5. The Effect of a Surface Active Medium on a Change of Effective Surface Energy of a Material

The carrying capacity of deformed solids in certain cases essentially depends on surrounding body of the working (surface active) medium. For a deformed solid with cracks this influence affects in the first place the change of effective surface energy of a given material. As P. A. Rebinder [147] first noted, the surface active medium affects deformation and rupture of solids mainly in the environment of the sharp (dead-end) ends of the developed cracks. In just these regions of a deformable body the adsorptive influence of the

medium causing the change of γ leads to a change of the strength properties of the solid as a whole.

Consequently, by placing samples in different media and determining in every specific case γ, the influence of these media on strength properties of the material of samples under a quasi-static load can be estimated. Such research in study of the influence of certain surface active media on a change of effective surface energy of silicate glass was conducted in works [127, 128].

By the method described in Sections 2 and 3 of this chapter, the values of effective surface energy were determined for silicate glass in dry air and in a surface active medium (water, methyl alcohol, vaseline oil) at room temperature. The chemical composition of the glass was following (%): 72.7 SiO_2; 1.45 Al_2O_3; 7.6 CaO; 3.73 MgO; 14.05 Na_2O; 0.1 Fe_2O_3; 0.37 SO_3.

The experimental plates cut from a glass sheet in the delivered state, had the dimensions: $2L \times a = 36 \times 22$ cm (see Fig. 18). In the center of every plate a small hole approximately 8 mm in diameter was drilled, across the diameter of the hole was made a through initial crack of length $2l < \frac{L}{2}$. The prepared plate samples were loaded through the hole by rods 7 and 8 (see Fig. 22). For every value of l forces P_*, were determined, then by formula (II.6) γ was calculated.

On Fig. 31 is shown the dependence of force P on the length of the crack for the case when the plate was tested first in dry air at room temperature (Fig. 31, curve 1), and then in water at room temperature (curve 2), after which it was dried at a temperature of 30°C for 5 h and again tested at room temperature in dry air (curve 3). For the material of this plate the mean value of effective surface energy of silicate glass in dry air and in water respectively was $2 \cdot 10^{-4}$ and $1{,}5 \cdot 10^{-4}$ J/cm^2. Twelve plates were similarly tested. Mean values of effective surface energy all over the plates was the following: in air $\gamma = 2{,}4 \cdot 10^{-4}$ J/cm^2, in water $\gamma = 1{,}8 \cdot 10^{-4}$ J/cm^2.

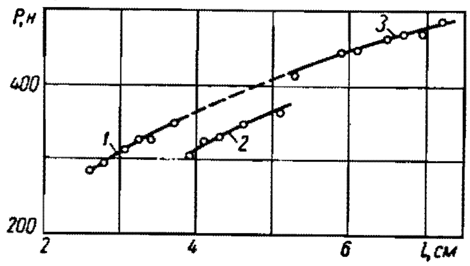


Fig. 31.

In this way the influence of methyl alcohol and vaseline oil on a change of γ was investigated. As a result of these investigations it has been established [127, 128] that when glass is immersed in water the latter as a very surface active medium [1, 2] with respect to silicate glass essentially (approximately by 25%) decreases γ, methyl alcohol — approximately by 15%, and vaseline oil does not lower γ. Obtained data correspond to results of work [18].

6. Estimating the Tendency of a Metal to Cold-Shortness

Effective surface energy γ, as already was noted, is the sum of the true surface energy (γ_0) of a given material and the energy of plastic deformation (γ_n) in the thin layer of material adjacent to the surface of the crack. On the whole energy γ can be treated just as the characteristic (measure) resistivity of a material to the propagation of a rupture crack in it. This follows in particular from formulas for determination of limit loads for a deformable solid with macrocracks (see, for example, formula (II.8). According to these formulas the greater the γ of a given material under preassigned conditions of its deformation, the more difficult it is to realize propagation of a crack in it. This suggests an idea that for deformable metals when temperature is reduced (or in the presence of other factors promoting a decrease of γ_n) we will obtain a decrease of the limit load, i.e., the load causing crack propagation in a body. Consequently, if one were to construct the graph of the dependence of γ of a given metal on temperature (its reduction), then its temperature treshold of cold-shortness (when, for example, a metal is inclined to cold-shortness)

can be characterized by a certain temperature $T = T_{\text{кр}}$, starting from which $(T \ll T_{\text{кр}})$ there is an essential decrease of $\gamma(T)$.

Thus, an estimate of the inclination of a metal to brittle rupture with a decrease of temperature, i.e., determination of the temperature of cold-shortness of a metal, can be obtained in the following way. From a given metal it is necessary to prepare sample plates with cracks, to rupture the samples following the pattern shown on Fig. 18 and to determine for each (lowered) test temperature T the limit load $P = P_*$, and then by formula (II.9) to calculate the value $\gamma = \gamma(T)$. According to obtained data, it is possible to construct the graph of the change of $\gamma(T)$. Using this graph, temperature $T = T_{\text{кр}}$, can be found at which there is a sharp change of values $\gamma(T)$. This temperature is the temperature threshold of cold-shortness for a given metal. If on the graph $\gamma(T)$ there is no characteristic point $T = T_{\text{кр}}$, this means that the metal is not inclined in the given interval of temperatures to cold-shortness.

On the basis of this procedure the author and S. Ye. Kovchik[2] determined the temperature of cold-shortness for U8 steel. From steel sheets were cut sample plates 100 × 120 × 2.5 mm. In the center of every plate a hole was drilled to connect to plate the clamps of the rupture-test machine. Further the samples were subjected to heat treatment: quenching at a temperature of 810°C in oil with subsequent low-temperature tempering at T = 150°C.

To create the initial crack the sample was covered by bakelite varnish and dried. Then with a razor the varnish was scratched in the direction of the diameter of the hole and symmetric relative to its center (parallel to base of plate), as a result of which a narrow slot up to the metal was formed. The length of the slot corresponded to the desirable length of the initial crack in the plate. The prepared plate was put in a vessel with an electrolyte (25% aqueous solution of sulfuric acid) and saturated with hydrogen, joining to the plate the negative pole of the current rectifier (anode was platinum wire immersed in electrolytic solution). For 5-10 min through the plate a current of 2 A was passed, after which it was

removed from the solution, washed and extended until a crack of preassigned length formed.

After formation of the crack, samples were tempered at 400°C for 2 h to achieve the desirable hardness of the metal, as a result of which the hardness of the plate was approximately 45 HRC. Then with a microscope the length $(2l_0)$ of the initial crack was measured.

Sample plate 1 (Fig. 32) was placed in thermostat 2, attached to the rupture-test machine, and by means of rods 3 and 5 connected with the grips of this machine. To the tested plate at the tip of the crack was pressed copper-constantan thermocouple 4, in series with which was the control thermometer. For convenience of installation and fastening the samples, the side wall of the thermostat was detachable. When the plate sample was set in the thermostat and attached to the grips of the rupture-test machine, it was cooled to a particular temperature and held at this temperature 10-15 minutes. After that the plate was ruptured, and limit load $P = P_*$ was fixed. Thus, the experiment was carried out in a range from the temperature of liquid nitrogen (-196°C) to room temperature at intervals of 20 degrees.

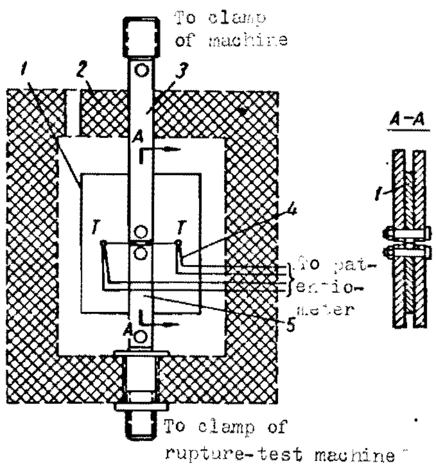


Fig. 32.

According to experimental data formula (II.9) was used to calculat $\gamma(T)$ for U8 steel with a hardness of approximately 45 HRC (Fig. 33a). It was established that a change of effective surface energy in the interval -196 to -40°C is insignificant. The surface of discontinuity of samples in this interval was a characteristic surface of brittle rupture.

With an increase of temperature near T = -40°C is observed a sharp change (increase) of values of $\gamma(T)$ as compared to values of $\gamma(T)$ when $T < -40°C$. A further increase of temperature in the interval -20 to +20°C insignificantly affects the change of $\gamma(T)$ as compared to $T < -20°C$. Experimental data show that the fracture surface of the samples in this interval of temperatures has a viscous character.

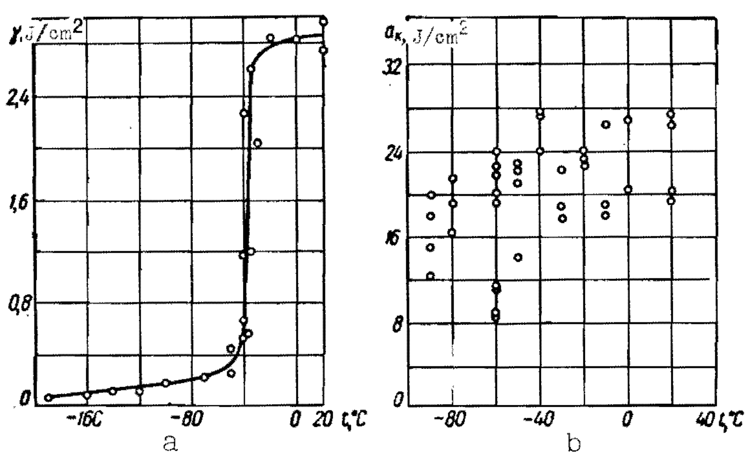


Fig. 33.

Thus, according to experimental data (Fig. 33a) for U8 steel with a hardness of approximately 45 HRC the temperature (threshold) of cold-shortness $T_{кр} = -40°C$.

For comparison an attempt was made to determine the threshold of cold-shortness during standard tests of cut samples on resilience. On Fig. 33b the points show the values of impact toughness a_k, determined for the investigated steel as a result of testing at reduced temperatures on an impact tester of the MK-30A type standard samples with cuts (cross section of samples 10 × 10 mm, depth of cut

2 mm, radius of fastening 1 mm). Experimental data show that for high-carbon steels near the threshold of cold-shortness there is a considerable scattering of impact toughness values. Consequently, on the basis obtained values of a_k it is difficult to determine the cold-shortness temperature of this steel. By the change of $\gamma(T)$ with a decrease of the test temperature the cold-shortness temperature $(T = T_{кр})$ can be determined well enough.

In conclusion let us note that study of the change of effective surface energy at increased rates of deformation of metal and elevated temperatures, if the mechanism of destruction is close to quasi-brittle, also presents interest.

Footnotes

[1]Engineers B. T. Dyachenko and S. T. Baranovich participated in preparation of the NaCl samples and carrying out the experiment.

[2]Panasyuk V. V., Kovchik S. Ye. — Plant laboratory, 1967, No. 4.

CHAPTER III

EXTENSION AND COMPRESSION OF PLATES WITH CRACKS

1. Basic Relationships of the Plane Theory of Elasticity. Kolosov-Muskhelishvili Formulas

The elastic equilibrium of solids is described by equations of the two-dimensional problem of the theory of elasticity[1] in the following cases of the stressed and deformed state of a body: 1) in the case of plane deformation of a body, i.e., during deformation of cylindrical or prismatic bodies of constant cross section, when the body is subjected to deformation by external forces normal to its axis and identical for all transverse (normal) sections of the body; 2) in the case of deformation of thin plates by forces acting in its plane ("generalized plane stress").

In such cases for determination of the stressed and deformed state in an arbitrary point of a deformable elastic isotropic body it is necessary to find three components of the stress tensor – σ_x, σ_y, τ_{xy} (Fig. 34) and three components of the deformation tensor – e_x, e_y, e_{xy}. For problems of the plane theory of elasticity these components are a function of only two variables: x and y, if system of cartesian coordinates xOy is selected so that it coincides either with the cross section of the rod, or with the middle plane of the plate.

Components of the deformation tensor in an arbitrary point of a deformable body are expressed through projections u and v of the

vector of displacements of this point by Cauchy formulas:

$$e_x = \frac{\partial u}{\partial x}; \quad e_y = \frac{\partial v}{\partial y}; \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{III.1}$$

where u and v are component of the displacement vector accordingly in the direction of the x- and y-axis; $u = u(x, y)$; $v = v(x, y)$.

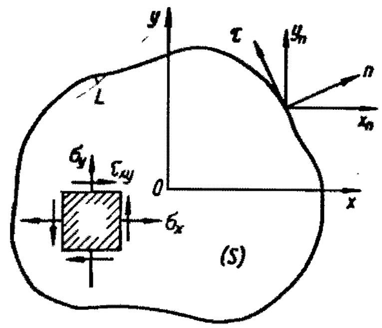


Fig. 34.

In works of G. V. Kolosov [60] and N. I. Muskhelishvili [103] it is shown that components of the stress tensor (σ_x, σ_y, τ_{xy}) and projections of the vector of displacements (u, v) in the two-dimensional problem of the theory of elasticity are determined through two analytic functions – $\phi_1(z)$ and $\psi_1(z)$:

$$\sigma_x + \sigma_y = 2\,[\varphi_1'(z) + \overline{\varphi_1'(z)}] = 2\,[\Phi(z) + \overline{\Phi(z)}]; \tag{III.2}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\,[\bar{z}\varphi_1''(z) + \psi_1'(z)] = 2\,[\bar{z}\Phi'(z) + \Psi(z)]; \tag{III.3}$$

$$\Phi(z) = \frac{d\varphi_1(z)}{dz} = \varphi_1'(z), \quad \Psi(z) = \frac{d\psi_1(z)}{dz} = \psi_1'(z);$$

$$2G(u + iv) = \varkappa\varphi_1(z) - z\overline{\varphi_1'(z)} - \overline{\psi_1(z)}. \tag{III.4}$$

Here z is a complex variable in plane xOy, $z = x + iy$; $\bar{z} = x - iy$; κ – constant expressed through Poisson factor ν, for plain strain $\kappa = 3 - 4\nu$; for the generalized plane stressed state $\kappa = \frac{3 - \nu}{1 + \nu}$; G – shear modulus, $G = \frac{E}{2(1 + \nu)}$, where E – Young's modulus.

Functions $\phi_1(z)$ and $\psi_1(z)$ are satisfied by the following boundary conditions.

1. In the case of the first basic problem, i.e., when on contour L (see Fig. 34) of domain S external stresses are assigned, we have

$$\varphi_1(t) + t\overline{\varphi_1'(t)} + \overline{\psi_1(t)} = i\int_0^s (X_n + iY_n)\,ds = f_1(t) + if_2(t), \qquad \text{(III.5)}$$

where X_n and Y_n are assigned projections of external stresses acting on contour L, as functions of arc s, $X_n = X_n(s)$, $Y_n = Y_n(s)$ (index "n" designates that shown quantities pertain to a area with external normal n (see Fig. 34); $f_1(t)$, $f_2(t)$ – known functions; t – affix of point of contour L of domain S.

2. In the case of the second basic problem, i.e., when on contour L of domain S shifts are assigned, boundary condition have the form

$$\varkappa\varphi_1(t) - t\overline{\varphi_1'(t)} - \overline{\psi'(t)} = 2G\,[q_1(t) + iq_2(t)], \qquad \text{(III.6)}$$

where $q_1(t)$ and $q_2(t)$ – known functions $[q_1(t) = u(t);\ q_2(t) = v(t)]$ on contour L. If boundary conditions are assigned on contour L of domain S, i.e., the right side of equation (III.5) or (III.6) is assigned, and functions $\phi_1(z)$ and $\psi_1(z)$ are found analytic in domain S, satisfying on the contour of the domain boundary condition (III.5) or (III.6), the problem about elastic equilibrium of domain S thereby is solved. Indeed, if function $\phi_1(z)$ and $\psi_1(z)$ or $\Phi(z)$ and $\Psi(z)$ are known, by formulas (III.2)-(III.4) one can determine the components of the stress tensor and the vector of displacements in any point of domain S.

On the basis of formulas (III.2) and (III.3) it is easy to verify that the stress condition in a body (in domain S) will not be changed if function $\phi_1(z)$ is replaced by $\phi_1(z) + Ciz + C_0$, and function $\psi_1(z)$ by $\psi_1(z) + C_0'$, where C and C_0, C_0' are respectively real and complex constants. Further, using formula (III.4), it is possible to show that components $Ciz + C_0$ and C_0' characterize the shift of a body as a rigid whole. Consequently, these constants can be in a determined way fixed [103].

Stresses and displacements are simple, therefore analytic functions $\phi_1(z)$ and $\psi_1(z)$ in a finite simply connected domain also can be simple and regular. For an unbounded domain (plate) with a hole, i.e., for a region consisting of all plane xOy, from which are removed finite parts of it (analogously for a finite double connected region), functions $\phi_1(z)$ and $\psi_1(z)$ have the following form:

$$\varphi_1(z) = -\frac{X+iY}{2\pi(1+\varkappa)}\ln z + (B+iC)z + \varphi_0(z) \quad (\Gamma = B+iC); \tag{III.7}$$

$$\psi_1(z) = \frac{\varkappa(X+iY)}{2\pi(1+\varkappa)}\ln z + (B'+iC')z + \psi_0(z) \quad (\Gamma' = B'+iC'), \tag{III.8}$$

where X and Y are components of the main vector of external stresses applied to the boundaries of the examined region; B, B', C and C' – real constants; $\phi_0(z)$ and $\psi_0(z)$ – holomorphic functions in the environment of a point at infinity.

Constants B + iC and B' + iC' are determined from conditions of tension in a point at infinity of a deformable body:

$$\lim_{|z|\to\infty}(\sigma_x+\sigma_y) = 4B \text{ и } \lim_{|z|\to\infty}(\sigma_y-\sigma_x+2i\tau_{xy}) = 2(B'+iC').$$

Designating by p and q ($p \geq q$) values of main stresses in points at infinity of the plate, and by α the angle between axis Ox and direction of stresses p, we obtain

$$\operatorname{Re}\Gamma = B = \frac{p+q}{4};\ \Gamma' = B'+iC' = -\frac{1}{2}(p-q)e^{-2i\alpha}. \tag{III.9}$$

The size of constant C is caused by the rotation of an infinitely remote part of the plane and does not affect the distribution of stresses. Subsequently we will assume C = 0.

If $z = \omega(\zeta)$ is a rational function conformally mapping the exterior (or interior) of unit circle $|\zeta| \geq 1$ in parametric plane ζ onto domain S, occupied by the body, formulas (III.2)-(III.6) can be transformed to new variable $\zeta = \rho e^{i\nu}$:

$$\left.\begin{aligned}\sigma_\rho+\sigma_\nu&=2\,[\Phi(\zeta)+\overline{\Phi(\zeta)}];\\ \sigma_\nu-\sigma_\rho+2i\tau_{\rho\nu}&=\frac{2\zeta^2}{\rho^2\overline{\omega'(\zeta)}}\,[\overline{\omega(\zeta)}\,\Phi'(\zeta)+\omega'(\zeta)\,\Psi(\zeta)];\\ 2G(u+iv)&=\varkappa\varphi(\zeta)-\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\,\overline{\varphi'(\zeta)}-\overline{\psi(\zeta)},\end{aligned}\right\}\qquad\text{(III.10)}$$

where

$$\varphi(\zeta)=\varphi_1[\omega(\zeta)],\quad \psi(\zeta)=\psi_1[\omega(\zeta)],$$
$$\Phi(\zeta)=\frac{\varphi'(\zeta)}{\omega'(\zeta)},\quad \Psi(\zeta)=\frac{\psi'(\zeta)}{\omega'(\zeta)}\ ;$$

σ_ρ, σ_ν, $\tau_{\rho\nu}$ – components of the stress tensor in curvilinear system (ρ, ν) of orthogonal coordinates. These components coincide correspondingly with components σ_x, σ_y and τ_{xy} under the condition that a moving linear system of cartesian coordinates with its origin in the examined point is oriented so that axis Oy is tangent to the curve ρ = const at the given point.

Boundary conditions (III.5) and (III.6) in the transformed region, when variable $\zeta(z = \omega(\zeta))$ is introduced, have the form

$$\varphi(\sigma)+\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi'(\sigma)}+\overline{\psi(\sigma)}=f_1+if_2\quad\text{when}\quad|\zeta|=1$$

and (III.11)

$$\varkappa\varphi(\sigma)-\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi'(\sigma)}-\overline{\psi(\sigma)}=2G(g_1+g_2)\quad\text{when}\quad|\zeta|=1,$$

where $|\zeta| = 1$ is the circumference of a unit radius in plane ζ, which corresponds to contour L of the examined domain S; $\sigma = e^{i\nu}$ – arbitrary point on circumference $|\zeta| = 1$; functions f_1 and f_2, characterizing external stresses on the boundary of domain S, are known functions of variable t, but not $t = \omega(\sigma)$, therefore f_1 and f_2 in (III.11) are known functions of variable σ; analogously to this in the case of the second basic problem functions g_1 and g_2 are also known functions of σ.

During conformal mapping of $z = \omega(\zeta)$ functions $\phi(\zeta)$ and $\psi(\zeta)$ have the same structure with respect to the variable ζ as functions $\phi_1(z)$ and $\psi_1(z)$ with respect to the variable z.

2. Determination of Limit Stresses for an Infinite Plate with Two Collinear Cracks

Resolution of the problem about maximum equilibrium of a plate with two collinear cracks of unequal length and the setting up of the working formulas is found in work [125]. Systems of collinear cracks of equal length located along a line in the elastic plane are examined in works [20, 85].

Two cracks of unequal length. Let us consider an infinite elastic plane (plate) with two macroscopic cracks of unequal length, located along one line. Let us introduce system of cartesian rectangular coordinates xOy is such a way that axis Ox (Fig. 35) will coincide with the line of the cracks, and designate by -d, -c, a, b the abscissas of points of ends of the cracks. The thickness of the plate is taken as unity.

Let us assume that such a plate on infinity is extended by monotonically increasing external stresses p. Then the minimum value of stress after which the propagation (increase of length) of at least one of these cracks is possible (Fig. 35), will be p_*.

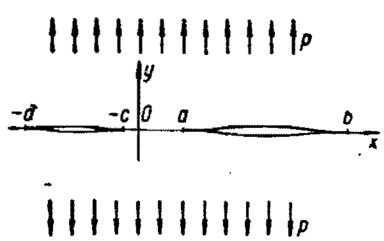


Fig. 35.

If by $p_*^{(a)}$, $p_*^{(b)}$, $p_*^{(c)}$, $p_*^{(d)}$ we designate stresses $\sigma_y = p$ at which the propagation of cracks goes in the direction of the abscissas respectively a, b, -c, -d, then we obtain

$$p_* = \min\{p_*^{(a)}, p_*^{(b)}, p_*^{(c)}, p_*^{(d)}\}. \qquad \text{(III.12)}$$

In accordance with the results of Section 6, Chapter I for determination of stresses $p_*^{(a)}$, $p_*^{(b)}$, $p_*^{(c)}$, $p_*^{(d)}$ we have the following equation:

$$\lim_{x \to (x_j \pm 0)} \{\pm \sqrt{|x - x_j|}\, \sigma_y^*(x, 0)\} = \frac{K}{\pi} \qquad (j = 1, 2, 3, 4), \tag{III.13}$$

where x_j is the abscissa of the end of the crack (correspondingly a, b, -c, -d); $\sigma_y^*(x, 0)$ – component of tensor of elastic stresses in the plane y = 0 (for our problem).

Thus, to solve the formulated problem in the beginning it is necessary to determine stresses $\sigma_y(x, 0)$. Under conditions of a two-dimensional problem components of the elastic stress tensor can be determined by formulas (III.2) and (III.3) if the function of stresses $\Phi(z)$ and $\Psi(z)$ are known. For our problem these functions have the following form [103];

$$\begin{gathered} \Phi(z) = \frac{c_0 z^2 + c_1 z + c_2}{\sqrt{(z-a)(z-b)(z+c)(z+d)}} - \frac{1}{2}\overline{\Gamma}' \qquad (z = x + iy); \\ \Psi(z) = \overline{\Omega}(z) - \Phi(z) - z\Phi'(z) \qquad (\Omega(z) = \Phi(z) + \overline{\Gamma}'), \end{gathered} \tag{III.14}$$

where $\overline{\Omega}(z)$ – function taking values conjugate with $\Omega(z)$ in points $\bar{z} = x - iy$, i.e., in points constituting a mirror image of points z in the real axis; $\Gamma' = \frac{1}{2}p$; $c_0 = \frac{1}{2}p$; coefficients c_1 and c_2 are determined from conditions of the uniqueness of shifts [103]. For our problem we have

$$\begin{gathered} \int_{-d}^{-c} \frac{c_0 x^2 + c_1 x + c_2}{\sqrt{(b-x)(x+d)(x-a)(x+c)}}\, dx = 0; \\ \int_{a}^{b} \frac{c_0 x^2 + c_1 x + c_2}{\sqrt{(b-x)(x+d)(x-a)(x+c)}}\, dx = 0. \end{gathered} \tag{III.15}$$

On the basis of formulas (III.2), (III.3) and (III.4) we obtain

$$\begin{aligned} \sigma_y(x, 0) &= \frac{2(c_0 x^2 + c_1 x + c_2)}{\sqrt{(x-a)(x-b)(x+c)(x+d)}} = \\ &= \frac{p(x^2 + A_1 x + A_2)}{\sqrt{(x-a)(x-b)(x+c)(x+d)}}. \end{aligned} \tag{III.16}$$

Here x takes values corresponding to points outside the crack, and coefficients A_1 and A_2 are expressed so:

$$A_1 = \frac{c_1}{c_0}; \quad A_2 = \frac{c_2}{c_0}.$$

According to equalities (III.15):

$$\begin{gathered}A_1 = -\frac{1}{2}(a + b - c - d);\\ A_2 = \frac{1}{2}(a + b + 3d - c)(b + d)\frac{\Pi(n, k)}{F(k)} - \\ -(b + d)^2\frac{I_2(n, k)}{F(k)} - \frac{1}{2}d(a + b + d - c).\end{gathered} \tag{III.17}$$

Here F(k), Π(n, k) are full elliptic integrals of type I and III with modulus k;

$$I_2(n, k) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{(1 + n\sin^2\varphi)^2\sqrt{1 - k^2\sin^2\varphi}}, \tag{III.18}$$

where modulus k and parameter n are expressed through values of abscissas of the ends of the crack:

$$k^2 = \frac{(b - a)(d - c)}{(b + c)(d + a)}; \quad n = \frac{b - a}{d + a}. \tag{III.19}$$

Putting expression (III.16) in equality (III.13) and carrying out passage to the limit as $x \to x_j$ for each end of a crack, we obtain the formula for $p_*^{(a)}$, $p_*^{(b)}$, $p_*^{(c)}$ and $p_*^{(d)}$. Thus, for the ends of a crack (a, 0) and (b, 0) accordingly we have

$$\begin{aligned}|x - x_j| = a - x \quad &\text{when} \quad x \to a - 0;\\ |x - x_j| = x - b \quad &\text{when} \quad x \to b + 0.\end{aligned}$$

On the basis of these equalities and formulas (III.13) and (III.16) we obtain

$$\begin{gathered}p_*^{(a)} = -\frac{K}{\pi}\cdot\frac{\sqrt{(b - a)(a + c)(a + d)}}{a^2 + A_1 a + A_2};\\ p_*^{(b)} = \frac{K}{\pi}\cdot\frac{\sqrt{(b - a)(b + c)(b + d)}}{b^2 + A_1 b + A_2}.\end{gathered} \tag{III.20}$$

where coefficients A_1 and A_2 are determined by formulas (III.17)-(III.19). Analogous formulas can be obtained for values of

$$p_*^{(c)} \text{ and } p_*^{(d)}.$$

Two cracks of equal length. If in equalities (III.20) we set $c = a$, $d = b$, we obtain formulas for determination of limit load in the case of a plate weakened by two collinear cracks of equal length, when in points at infinity of the plate external tensile stresses p, are applied perpendicularly to the line of location of the cracks (Fig. 36).

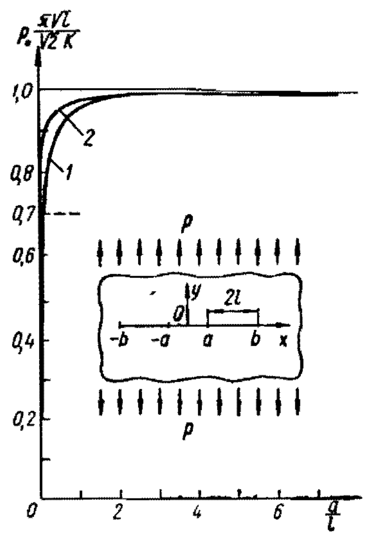


Fig. 36.

Simple working formulas require complex transformations of equalities (III.17)-(III.19) for $c = a$, $d = b$. However, such formulas can be obtained by a simpler method.

At $c = a$ and $d = b$ the state of strain in a plate is symmetric with respect to plane $x = 0$, therefore $\sigma_y(x, 0) = \sigma_y(-x, 0)$. Then on the basis of formula (III.16) we have $c_1 = 0$. Thus, for a plate with two collinear cracks of equal length (see Fig. 36) we obtain

$$\sigma_y(x, 0) = \frac{2(c_0x^2 + c_2)}{\sqrt{(x^2 - b^2)(x^2 - a^2)}}, \qquad \text{(III.21)}$$

where coefficient c_2 is determined according to equality (III.15) from the equation

$$\int_a^b \frac{c_0x^2 + c_2}{\sqrt{(b^2 - x^2)(x^2 - a^2)}} dx = 0 \qquad (c_1 = 0). \tag{III.22}$$

Considering in equation (III.22) that

$$x^2 = b^2 - (b^2 - a^2)\sin^2\varphi$$

and carrying out simple transformations, we find

$$c_2 = -b^2c_0\frac{E(e)}{F(e)} \quad \left(c_0 = \frac{1}{2}p\right), \tag{III.23}$$

where E(e) and F(e) are full elliptic integrals correspondingly of type II and I with modulus $e = \frac{\sqrt{b^2 - a^2}}{b}$,

$$F(e) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - e^2\sin^2\varphi}}; \quad E(e) = \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin^2\varphi}\,d\varphi.$$

Consequently, for our problem

$$A_1 = \frac{c_1}{c_0} = 0 \text{ and } A_2 = \frac{c_2}{c_0} = -b^2\frac{E(e)}{F(e)}. \tag{III.24}$$

Then on the basis of formulas (III.20) we obtain

$$p_*^{(a)} = \frac{be\sqrt{2a}F(e)}{b^2E(e) - a^2F(e)} \cdot \frac{K}{\pi}, \tag{III.25}$$

$$p_*^{(b)} = \frac{\sqrt{2}eF(e)}{\sqrt{b}[F(e) - E(e)]} \cdot \frac{K}{\pi}. \tag{III.26}$$

Formulas (III.25) and (III.26) are obtained by other means in work [123, 227].

It is useful to conduct an analysis of certain maximum cases emanating from formulas (III.25) and (III.26). Thus if a → 0, then e → 1

and $$\lim_{e\to 1}\frac{eF(e)}{F(e)-E(e)}=1,\quad \lim_{e\to 1}\frac{be\sqrt{a}F(e)}{b^2E(e)-a^2F(e)}=0.$$

When a = 0, i.e., for a plate with one crack of length 2b, from formulas (III.25) and (III.26) we obtain

$$p_*^{(a)}=0;\quad p_*^{(b)}=p_*^{(r)}=\frac{\sqrt{2}K}{\pi\sqrt{b}}. \qquad \text{(III.27)}$$

As expected, formula (III.27) coincides with the Griffith formula for one isolated crack of length 2b.

Let us examine another maximum case, when collinear cracks of equal length are located a considerable distance from each other, so that it is possible to consider $a \to \infty$ and $b \to \infty$, but $b - a = 2l =$ = const, where $2l$ is the length of an individual crack. In this case $e \to 0$ and formulas (III.25) and (III.26) take the form

$$p_*^{(a)}=p_*^{(b)}=\frac{\sqrt{2}K}{\pi\sqrt{l}}, \qquad \text{(III.28)}$$

i.e., in this case every crack behaves as an independent isolated crack of length $2l$.

On Fig. 36 are built curves of the change of stress $p_*^{(a)}$(1) and $p_*^{(b)}$(2) in accordance with formulas (III.25) and (III.26) depending upon the relationship $\lambda = \frac{a}{l}$. According to these curves, the values of limit load $p_*^{(a)}$ are always smaller than values of $p_*^{(b)}$, i.e., the development of two collinear cracks at first proceeds toward each other as a result of destruction of the connector, then (after they merge) one crack of length 2b appears.

If connector 2a between the cracks is rather small as compared to length $2l$ of an individual crack, i.e., when $\frac{a}{l} < \lambda_*$, where λ_* is a certain critical value of the ratio $\frac{a}{l}$, then the destruction of the connector (external stresses p achieve level $p_*^{(a)}$) still does not involve the rupture of the entire plate. In this case the limit load for the plate is determined by the Griffith formula, when the length

of the crack is 2b, i.e., formula (III.27) for $p_*^{(b)}$. When $\frac{a}{l} > \lambda_*$ the destruction of the connector involves the rupture of the entire plate.

On the basis of formulas (III.25) and (III.27) for determination of parameter $\lambda_* = \frac{a_*}{l}$ we have equation

$$p_*^{(a)} = p_*^{(r)} \text{ or } \frac{be_* \sqrt{a_*} F(e_*)}{b^2 E(e_*) - a_*^2 F(e_*)} = \frac{1}{\sqrt{b}}. \tag{III.29}$$

Considering that

$$\lambda_* = \frac{a_*}{l} = \frac{2a_*}{b - a_*}, \quad b = a_* \frac{2 + \lambda_*}{\lambda_*} \text{ and } e_* = \frac{2\sqrt{1 + \lambda_*}}{2 + \lambda_*}, \tag{III.30}$$

equation (III.29) can be written as:

$$f(\lambda_*) = \frac{2\sqrt{\lambda_* (1 + \lambda_*)(2 + \lambda_*)} F(e_*)}{(2 + \lambda_*)^2 E(e_*) - \lambda_*^2 F(e_*)} = 1. \tag{III.31}$$

Solving equation (III.31) relative λ_*, we find

$$\lambda_* \approx 0{,}11. \tag{III.32}$$

Thus, limit load $p = p_{**}$ for a plate with two collinear cracks of equal length, extended to infinity by monotonically increasing stresses (see Fig. 36), is determined by the formula

$$p_{**} = \begin{cases} p_*^{(a)} & \text{when } \frac{a}{l} > \lambda_*, \\ p_*^{(r)} & \text{when } \frac{a}{l} \leqslant \lambda_*, \end{cases} \tag{III.33}$$

where $p_*^{(a)}$ and $p_*^{(r)}$ are calculated correspondingly by formulas (III.25) and (III.27).

If the connector between cracks is such that $\frac{a}{l} > 3$ (Fig. 36), the crack can practically be considered isolated (not affecting one another), and the limit load can be determined by formula (III.28).

3. Maximum Equilibrium of a Plate with Arbitrarily Oriented Linear or Curvilinear Crack

Above we examined problems about the limiting equilibricum state of a plate with linear cracks, when the plane of the cracks coincides with the plane of symmetry of the field of external stresses. In such cases, obviously, crack propagation in an isotropic body is directed along the plane of symmetry of external stresses and, as is shown in Chapter I, the external load becomes limit equilibrium, if condition (I.47) (for macroscopic cracks) holds.

Below the problem about limiting equilibrium of a plate with a macrocrack, when the plane of the crack does not coincide with the plane of symmetry of external stresses.

Let us consider an infinite plate weakened by a linear (or curvilinear) macroscopic crack (cut), assuming that the edges of the crack are free from external stresses and in points at infinity of the plate in mutually perpendicular directions are applied monotonically increasing external tensile stresses p and q. Let us assume that the material of the plate obeys Hooke's law up to the instant of rupture. It is necessary to determine values of stresses $p = p_*$ and $q = q_*$.

However, a determination of breaking points only by equation (I.47) is generally impossible because when the plane of the crack does not coincide with the plane of symmetry of the bending stresses, the direction of initial propagation of the crack is unknown beforehand, i.e., it is not known in what direction s in equation (I.47) approaches zero. In such a case to determine the initial direction of the crack propagation it is necessary to formulate additional conditions. If one were to introduce polar system of coordinates r, β with origin at the tip of the crack and polar axis directed along the tangent to the sides of the crack (Fig. 37a), then the component of breaking elastic stresses $\sigma_\beta(r, \beta)$ in the environment of the tip of the crack can be represented as:

$$\sigma_\beta(r, \beta) = \frac{N_0}{\sqrt{r}} + 0(1),$$

where N_0 – coefficient of intensity of stresses $\sigma_\beta(r, \beta)$ in the neighborhood of the examined tip, which depends on the acting loads, configuration of body, form of crack and angle β; $0(1)$ – limited part of stress component $\sigma_\beta(r, \beta)$ when $r \to 0$.

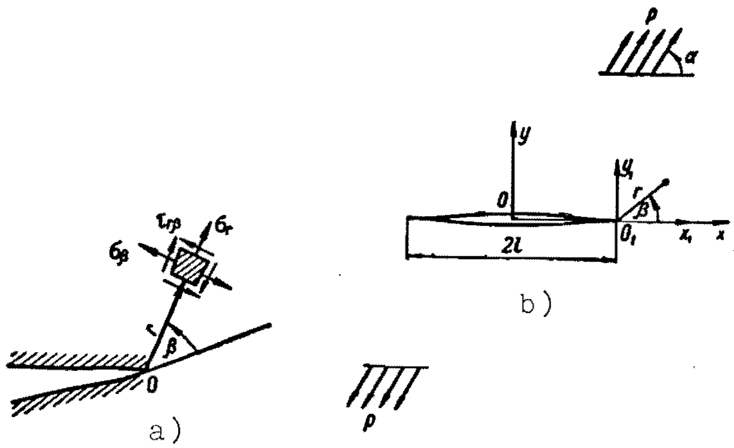


Fig. 37.

Having this in mind, let us introduce the following hypothesis: *initial crack propagation occurs in the plane, for which normal breaking stresses* σ_β *have a maximum value of the coefficient of intensity.*[2]

Thus, on the basis of the accepted hypothesis for determination of limit values of external stresses we obtain the equation

$$\lim_{r \to 0} \{\sqrt{r}\,\sigma^*_\beta(r, \beta_*)\} = \frac{K}{\pi}, \qquad \text{(III.34)}$$

where $\sigma^*_\beta(r, \beta)$ – stresses $\sigma_\beta(r, \beta)$ when $p = p_*$ and $q = q_*$, angle $\beta = \beta_*$, determining initial direction of crack propagation, is found in accordance with the accepted hypothesis from the equation

$$\lim_{r \to 0} \left\{ \sqrt{r}\, \frac{\partial \sigma_\beta(r, \beta)}{\partial \beta} \right\}_{\beta = \beta_*} = 0. \qquad \text{(III.35)}$$

Thus, if under an assigned system of external loads the value of elastic stresses (or value of coefficient N_0) in the neighborhood of the tips of a macroscopic crack is established, then, using equations (III.34) and (III.35), it is possible to calculate the limit value of these loads.

4. Unidirectional Extension of a Plate with Randomly Oriented Linear Crack

Formulation and solution of problem. Let us assume that an unlimited isotropic plate with linear crack (cut) of length $2l$ (Fig. 37b) is extended by monotonically increasing external stress p, applied in points as infinity of the plate and directed at an angle α to the plane of location of the crack. Let us assume that the plate belongs to rectangular system of cartesian coordinates xOy, the crack is located along axis Ox when $-l \leq x \leq l$ (see Fig. 37b), and plane xOy coincides with the middle plane of the plate (thickness of plate is accepted to be unity).

For the formulated problem it is necessary to determine the limit value of stresses p. At first we find components σ_x, σ_y, τ_{xy} of the tensor of elastic stresses in the neighborhood of the ends of the examined crack. These components are expressed through complex potentials $\Phi(z)$ and $\Omega(z)$ [103]:

$$\sigma_x + \sigma_y = 2\,[\Phi(z) + \overline{\Phi(z)}] \qquad (z = x + iy,\ \bar{z} = x - iy); \tag{III.36}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\,[(\bar{z} - z)\,\Phi'(z) + \bar{\Omega}(z) - \Phi(z)], \tag{III.37}$$

where function $\Omega(z)$ is connected with function $\Psi(z)$ in formula (III.3) by the equality

$$\Psi(z) = \overline{\Omega}(z) - \Phi(z) - z\Phi'(z). \tag{III.38}$$

For the examined problem functions $\Omega(z)$ and $\Phi(z)$, defined in work [103], have the following form:

$$\Phi(z) = \frac{(2\Gamma + \bar{\Gamma}')z}{2\sqrt{z^2 - l^2}} - \frac{1}{2}\bar{\Gamma}'; \quad \Omega(z) = \Phi(z) + \bar{\Gamma}' \tag{III.39}$$

$$\left(\lim_{|z| \to \infty} z^{-1}\sqrt{z^2 - l^2} = 1\right),$$

where Γ and Γ' are constants characterizing the state of strain in points at infinity of the plate. These constants, expressed by equations (III.9), for the examined problem have the form

$$\Gamma = \frac{1}{4}p, \quad \Gamma' = -\frac{1}{2}pe^{-2i\alpha}. \tag{III.40}$$

In order to determine stresses in the neighborhood the ends of a crack, it is advisable to cross from system of coordinates xOy to system of coordinates $x_1O_1y_1$ (see Fig. 37b) with origin at the tip of the crack. At the transfer of the origin of coordinates from point O to point O_1, i.e., at the transition to a new system of coordinates $x_1O_1y_1$ functions $\Phi_1(z_1)$ and $\Omega_1(z_1)$, playing the same role in system of coordinates $x_1O_1y_1$ as functions $\Phi(z)$ and $\Omega(z)$ in system of coordinates xOy, can be determined by known formulas [103]:

$$\Phi_1(z_1) = \Phi(z_1 + z_0); \quad \Omega_1(z_1) = \Omega(z_1 + z_0) + (\bar{z}_0 - z)\overline{\Phi'}(z_1 + z_0). \tag{III.41}$$

If one were to introduce polar system of coordinates r, β with origin at point O_1, taking as the polar axis the tangent to the sides of the crack in this point, then components σ_r, σ_β, $\tau_{r\beta}$ of the stress tensor in the polar system of coordinates can be expressed through functions $\Phi_1(z_1)$ and $\Omega_1(z_1)$ as [103]:

$$\sigma_r + \sigma_\beta = 2\left[\Phi_1(z_1) + \overline{\Phi_1(z_1)}\right] \quad (z_1 = re^{i\beta}); \tag{III.42}$$

$$\sigma_\beta - \sigma_r + 2i\tau_{r\beta} = 2\left[(\bar{z}_1 - z_1)\Phi_1'(z_1) + \bar{\Omega}_1(z_1) - \Phi_1(z_1)\right]e^{2i\beta}. \tag{III.43}$$

On the basis of formulas (III.39)-(III.43) we have

$$\sigma_r + \sigma_\beta = p\,\mathrm{Re}\left\{(1 - e^{2i\alpha})\frac{z_1 + l}{\sqrt{2lz_1 + z_1^2}}\right\} + p\cos 2\alpha;$$

$$\sigma_\beta - \sigma_r + 2i\tau_{r\beta} = p\left\{\frac{1}{2}(1 - e^{2i\alpha})\frac{(z_1 - \bar{z}_1)\,l^2}{\sqrt{(2lz_1 + z_1^2)^3}} + \right.$$
$$\left. + i\sin 2\alpha\frac{z_1 + l}{\sqrt{2lz_1 + z_1^2}} - \cos 2\alpha\right\}e^{2i\beta}. \qquad \text{(III.44)}$$

We will examine a small region in the neighborhood of the ends of the crack, i.e., we will examine the totality of points for which the inequality $|z_1| \ll l(r \ll l)$ is valid. For such a region we can write:

$$\frac{z_1 + l}{\sqrt{2z_1 l + z_1^2}} = \frac{1}{\sqrt{2}}\sqrt{\frac{l}{z_1}} + 0\left(\sqrt{\frac{z_1}{l}}\right);$$
$$\frac{(z_1 - \bar{z}_1)\,l^2}{\sqrt{(2lz_1 + z_1^2)^3}} = \frac{1}{2\sqrt{2}}\left(1 - \frac{\bar{z}_1}{z_1}\right)\sqrt{\frac{l}{z_1}} + 0\left(\sqrt{\frac{z_1}{l}}\right). \qquad \text{(III.45)}$$

Placing expression (III.45) in (III.44), after simple transformations we determine components of the stress tensor in the neighborhood of the ends of the crack, i.e., at $z_1 = re^{i\beta}$, $r \ll l$:

$$\sigma_\beta = \frac{1}{4\sqrt{2r}}\left[k_1\left(3\cos\frac{\beta}{2} + \cos\frac{3\beta}{2}\right) - 3k_2\left(\sin\frac{\beta}{2} + \sin\frac{3\beta}{2}\right)\right] +$$
$$+ p\sin^2\beta\cos 2\alpha + 0\left(\sqrt{\frac{r}{l}}\right); \qquad \text{(III.46)}$$

$$\sigma_r = \frac{1}{4\sqrt{2r}}\left[k_1\left(5\cos\frac{\beta}{2} - \cos\frac{3\beta}{2}\right) + k_2\left(-5\sin\frac{\beta}{2} + \sin\frac{3\beta}{2}\right)\right] +$$
$$+ p\cos^2\beta\cos 2\alpha + 0\left(\sqrt{\frac{r}{l}}\right); \qquad \text{(III.47)}$$

$$\tau_{r\beta} = \frac{1}{4\sqrt{2r}}\left[k_1\left(\sin\frac{\beta}{2} + \sin\frac{3\beta}{2}\right) + k_2\left(\cos\frac{\beta}{2} + 3\cos\frac{3\beta}{2}\right)\right] -$$
$$- p\sin\beta\cos\beta\cos 2\alpha + 0\left(\sqrt{\frac{r}{l}}\right). \qquad \text{(III.48)}$$

Here $0\left(\sqrt{\frac{r}{l}}\right)$ is a component part of the component of the stress tensor, which has the order $\left(\frac{r}{l}\right)^{\frac{1}{2}}$ $(r \ll l)$;

$$k_1 = p\sqrt{l}\sin^2\alpha, \quad k_2 = p\sqrt{l}\sin\alpha\cos\alpha,$$

where k_1, k_2 are coefficients of the intensity of stresses or stress concentration factors for the tip of the crack, respectively for symmetric and asymmetric distribution of stresses through angle β. These coefficients depend on loads applied to the body, form of the body and the crack.

Formulas analogous to formulas (III.46)-(III.48) can also be obtained by other means [219, 226].

Using formulas (III.34), (III.35) and (III.46) it is easy to find the relationship for determination of limit stresses $p = p_*$:

$$\frac{p_*}{4}\sqrt{\frac{l}{2}}\left\{\sin^2\alpha\left[3\cos\frac{\beta_*}{2} + \cos\frac{3\beta_*}{2}\right] - 3\sin\alpha\cos\alpha\left[\sin\frac{\beta_*}{2} + \sin\frac{3\beta_*}{2}\right]\right\} = \frac{K}{\pi}; \tag{III.49}$$

$$k_1\left[\sin\frac{\beta_*}{2} + \sin\frac{3\beta_*}{2}\right] + k_2\left[\cos\frac{\beta_*}{2} + 3\cos\frac{3\beta_*}{2}\right] = 0. \tag{III.50}$$

Solving equation (III.50) relatively to β_*, we find those values of angle β_* at which stresses $\sigma_\beta(r, \beta)$ attain the highest possible intensity. Values of angle β_* are determined by the following formula:

$$\beta_* = 2\,\mathrm{arctg}\,\frac{1 \mp \sqrt{1 + 8n^2}}{4n}, \tag{III.51}$$

where

$$n = \frac{k_2}{k_1} = \mathrm{ctg}\,\alpha. \tag{III.52}$$

In formula (III.51) the "+" corresponds to values of $k_1 < 0$, the "-" to value of $k_1 > 0$.

Thus, in accordance with formulas (III.49)-(III.52) for the examined problem (see Fig. 37) we find

$$p_* = \frac{K\sqrt{2}}{\pi\sqrt{l}} \cdot \frac{1}{\cos^3\frac{\beta_*}{2}\sin^2\alpha\left[1 - 3\operatorname{ctg}\alpha\operatorname{tg}\frac{\beta_*}{2}\right]}, \qquad \text{(III.53)}$$

where $0 < \alpha \leq \frac{\pi}{2}$; angle β_* is determined by the formula

$$\beta_* = 2\operatorname{arctg}\frac{1 - \sqrt{1 + 8\operatorname{ctg}^2\alpha}}{4\operatorname{ctg}\alpha}. \qquad \text{(III.54)}$$

Considering in formulas (III.53) and (III.54) that $\alpha = \frac{\pi}{2}$, we obtain the Griffith formula for a plate with a linear crack when the plate is pulled by external stresses p, directed perpendicular to the line of location of the crack:

$$\beta_* = 0, \quad p_* = \frac{K\sqrt{2}}{\pi\sqrt{l}}.$$

In accordance with formula (III.54) on Fig. 38 the change of initial direction of crack propagation (change of angle β_* as a function of the orientation of the crack when $0 < \alpha \leq \frac{\pi}{2}$ is graphed). According to this graph, for the considered problem the initial direction of crack propagation is close to the direction forming a right angle with the line of action of external stresses.

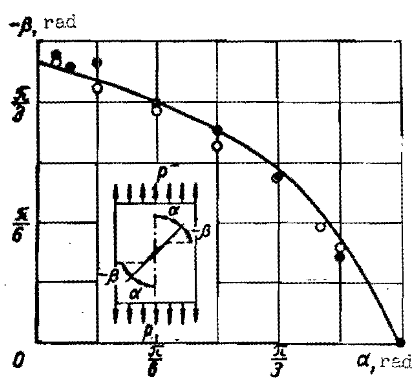


Fig. 38.

On Fig. 39 is shown the change of $\frac{\pi p_*\sqrt{l}}{\sqrt{2}K}$ depending upon angle $\alpha(0 < \alpha \leq \frac{\pi}{2})$. Using this figure or formulas (III.53 and (III.54), for the examined problem it is easy to find the value of limit stresses $p = p_*$ at a given value of angle $\alpha(0 < \alpha \leq \frac{\pi}{2})$. The minimum

value of $p_* = 0.97K\frac{\sqrt{2}}{\pi\sqrt{l}}$ (depending upon crack orientation) occurs at $\alpha \approx 1.19$ rad.

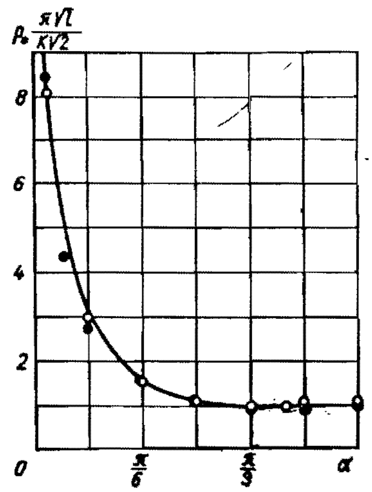


Fig. 39.

Experimental check of solution. Inasmuch as formula (III.54) is obtained on the basis of equality (III.35), which follows from the accepted hypothesis, the comparison of calculation results with experimental data is of considerable interest. For checking results of the calculation of limit values of $p = p_*$ and angle β_* according to formula (III.54) the following experiments were conducted [22, 132]. From a glass sheet (silicate and plastic) were cut plates whose dimensions are shown in Table 10 and 11, where a and 2L are respectively the longitudinal and transverse dimensions of the plate, and h is its thickness. In the center of every plate was drilled a small hole approximately 6 mm in diameter. With a glass cutter in the plate an initial linear cut was made passing through the center of the hole at a certain angle α to the longitudinal axis of the plate. This plate on special attachment with rigid rods was loaded through the hole perpendicularly to the line of the cut by monotonically increasing forces (see Fig. 18), as a result of which the cut turned into a crack through the entire thickness of the plate. Then length $2l$ of the initial crack and angle α between the plane of the crack and the longitudinal axis of the plate were measured. Thus groups of plates were prepared from silicate glass and plastic with

different direction of crack relatively to the longitudinal axis of the plate, i.e., with different angles α when $0 < \alpha \leq \frac{\pi}{2}$, and different crack lengths $2l$ (Table 10 and 11).

Table 10.

No. of [illegible]	a, cm	$2L$, cm	h, mm	l, mm	α, rad	P_*, daN	$\beta_*^{\text{л}}$, rad	$\beta_*^{\text{п}}$, rad	$\beta_*^{\text{ср}}$, rad	$P_*\sqrt{l_\alpha}$	$m(\alpha)$
1	25,0	24,9	2,2	44,0	$\frac{\pi}{2}$	132,5	0	0	0	1,61	1,01
2	23,8	17,9	1,7	35,5	$\frac{\pi}{2}$	81,4	0	0	0	1,57	0,99
3	23,8	17,8	1,7	34,0	$\frac{\pi}{2}$	84,8	0	0	0	1,61	1,01
4	23,8	17,8	1,7	33,4	$5\frac{\pi}{12}$	83,8	0,472	0,492	0,482	1,61	1,01
5	23,8	17,8	1,7	31,0	$5\frac{\pi}{12}$	85,3	0,485	0,357	0,421	1,58	0,99
6	25,0	15,0	2,1	32,5	$5\frac{\pi}{12}$	69,8	0,439	0,389	0,414	1,53	0,96
7	25,0	15,0	2,1	32,5	$5\frac{\pi}{12}$	84,8	0,363	0,347	0,355	1,54	0,97
8	25,0	15,0	2,1	33,0	$7\frac{\pi}{18}$	77,5	0,424	0,460	0,442	1,43	0,90
9	25,0	15,0	2,1	35,9	$7\frac{\pi}{18}$	90,7	0,597	0,523	0,560	1,72	1,08
10	23,8	17,8	1,7	34,8	$\frac{\pi}{3}$	91,2	0,755	0,714	0,734	1,76	1,10
11	23,8	17,8	1,8	31,5	$\frac{\pi}{3}$	83,8	0,606	0,719	0,662	1,48	0,92
12	25,0	15,0	2,0	36,3	$\frac{\pi}{3}$	71,2	0,723	0,760	0,741	1,41	0,88
13	23,8	17,8	1,8	34,9	$\frac{\pi}{4}$	95,7	0,788	0,853	0,820	1,76	1,12
14	23,8	17,8	1,9	37,3	$\frac{\pi}{4}$	112,8	0,939	0,925	0,932	2,01	1,26
15	23,8	17,8	1,8	36,0	$\frac{\pi}{4}$	110,4	0,830	0,809	0,820	2,07	1,30
16	25,0	17,9	2,0	56,5	$\frac{\pi}{6}$	114,7	0,933	1,064	0,998	2,39	1,50
17	25,0	18,0	2,2	55,6	$\frac{\pi}{6}$	144,2	1,119	1,078	1,098	2,69	1,69
18	25,0	18,0	2,2	80,2	$\frac{\pi}{6}$	98,1	0,922	1,001	0,962	2,27	1,42
19	30,0	18,0	2,2	92,7	$\frac{\pi}{12}$	224,5	1,065	1,018	1,042	5,42	3,40

Table 10 (Cont'd).

No. of plate	a, см	$2L$, см	h, мм	l, мм	α, rad	P_*, daN	$\beta_*^{\text{л}}$, rad	$\beta_*^{\text{п}}$, rad	$\beta_*^{\text{ср}}$, rad	$P_*\sqrt{l_\alpha}$	$m(\alpha)$
20	30,0	12,4	1,8	44,2	$\frac{\pi}{12}$	213,0	1,169	1,073	1,121	3,96	2.48
21	30,0	12,1	1,9	82,7	$\frac{\pi}{12}$	103,0	1,195	1,123	1,159	4.05	2,54
22	30,0	12,0	1,9	117,7	$\frac{\pi}{36}$	270,0	1,227	1,225	1,226	12,95	8,12

Table 11.

No. of plate	a, см	$2L$, см	h, мм	l, мм	α, rad	P_*, daN	$\beta_*^{\text{л}}$, rad	$\beta_*^{\text{п}}$, rad	$\beta_*^{\text{ср}}$, rad	$P_*\sqrt{l_\alpha}$	$m(\alpha)$
1	39	25	1,0	53,0	$\frac{\pi}{2}$	114,9	0	0	0	3,28	1
2	39	25	1,1	42,1	$5\frac{\pi}{12}$	131,0	0,398	0,392	0,395	3,00	0,92
3	39	25	1,1	43,9	$\frac{\pi}{3}$	129,0	0,661	0,779	0,720	3,08	0,94
4	44	25	1,1	41,2	$\frac{\pi}{4}$	168,0	0,944	0,916	0,930	3,82	1,16
5	37	25	1,0	45,2	$\frac{\pi}{6}$	183,5	1,013	1,070	1,042	4,93	1,51
6	48	25	1,1	62,9	$\frac{\pi}{12}$	285,0	1,213	1,226	1,220	8,37	2,55
7	34	16	1,0	120,8	$2\frac{\pi}{45}$	219,5	1,220	1,192	1,206	14,45	4,41
8	52	17	1,0	138,8	$\frac{\pi}{36}$	423,0	1,193	1,287	1,240	28,30	8,63

Such plate samples were stretched then in the direction of the longitudinal axis of the plate on a rupture-test machine of the MR-0.5 type with stretching rate of $6.6 \cdot 10^{-5}$ m/s. The samples were connected with the clamps of the machine by two methods: 1) drilling small holes in the upper and lower parts of the plate (far from the crack) and pressing the plate through these holes in the metal clamps; 2) gluing on the ends of the plates special leatherette strips with which the plate is fastened in the grips of the machine. Both methods of fastening samples within the limits of the accuracy of measurements gave identical results during the experiment.

During the extension of the plate with the crack directed at an angle α to its longitudinal axis, the force $Q = Q_*$ was fixed, after which crack propagation began and the plate was destroyed. On samples from silicate glass insignificant propagation (turn) of the initial crack was observed in the neighborhood of the ends $Q = (0.95-0.97)Q_*$; however, this occurred only when angle α was close to $\frac{\pi}{2}$. On plastic specimens initial crack propagation immediately led to complete rupture of the plate.

By the experimental values of maximum force Q_* the limit load was calculated by the formula

$$p_{*\alpha} = \frac{Q_*^{(\alpha)}}{2hL},$$

where h and 2L are correspondingly thickness and transverse dimension of the plate sample; the index α shows that the quantities pertain to a plate for which the linear crack is directed at an angle α to its longitudinal axis.

After the plate ruptures, the initial length of the crack $2l_\alpha$ for more percise determination was measured on horizontal comparator of the IZA-2 type. For every plate $p_{*\alpha}\sqrt{l\alpha}$ and the relation

$$m(\alpha) = \frac{p_{*\alpha}\sqrt{l_\alpha}}{p_{*\left(\frac{\pi}{2}\right)}\sqrt{l_{\frac{\pi}{2}}}} = \frac{\pi p_{*\alpha}\sqrt{l_\alpha}}{K\sqrt{2}}.$$

was calculated according to experimental measurements.

Average values of $m(\alpha)$ for every group of plates with cracks at the same value of angles α are represented on Fig. 39, where nonshaded circles correspond to silicate glass plates, and the shaded to plastic plates; the solid line shows the change of function $m(\alpha)$ according to the formula (III.53).

Furthermore, for every plate angle β_*, characterizing the initial direction of propagation of the initial crack is measured. Measurements were made out on a large instrument microscope of BMI-1 type.

Angle β_* (see Fig. 37) is the angle between the plane in which the initial crack is located and the tangent to the line of its initial propagation in the tip of the initial crack. During experimental determination of angles β_* it assumed that the tangent to the trajectory of initial crack propagation is the line passing through the tip of the initial crack and a point located on the path of initial propagation 0.3 mm from its beginning. On Fig. 40 is the dead-end region of crack $2l$ (at $\alpha = \pi/3$), magnified 30 times by a projection attachment of the instrument microscope.

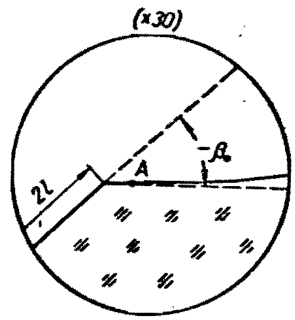


Fig. 40

In such a way for all ruptured plates the value of angles β_* were determined for both the left ($\beta_*^{Л}$) and right ($\beta_*^{П}$) end of the crack (Table 10 and 11). Mean values of angles $\beta_*^{cp} = \frac{1}{2}(\beta_*^{Л} + \beta_*^{П})$ for every group of plates are represented in the form of points on Fig. 38, where the nonshaded circles correspond to plates from silicate glass, and the shaded refer to plastic plates. The solid line is the dependence of angle β_* on α in accordance with formula (III.54).

Experimental data shown in Figs. 38 and 39 will agree well with results of theoretical analysis of the maximum-equilibrium state of a plate with arbitrarily oriented linear crack. Analogous dependence for angles β_* are established in work [185] for plastic plates weakened by arbitrarily oriented macrocracks.

5. Biaxial Extension-Compression of a Plate with Linear Crack and Construction of a Diagram of the Limit Stresses

Let us assume that an infinite isotropic plate of unit thickness is weakened by a linear crack of length $2l$ and is subjected to extension-compression in infinitely remote points by external stresses p and q, acting in mutually perpendicular directions, stresses p are directed at an angle α to the plane of the crack (Fig. 41). It is required to determine the value of limit stresses p_* and q_*.

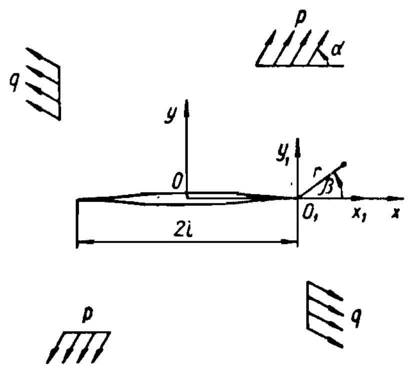


Fig. 41.

Equations (III.34) and (III.35) can be used. First let us find components of the elastic stress tensor in the neighborhood of the ends of the examined crack. We introduce rectangular system of cartesian coordinates xOy as is done in the preceding section and examine separately: 1) the case of an opening crack, i.e., when edges of the crack do not contact each other $\sigma_y(x, 0) = 0$ and $\tau_{xy}(x, 0) = 0$ and 2) the case when the edges of the crack touch $\sigma_y(x, 0) \neq 0$, $\tau_{xy}(x, 0) \neq 0$.

Opening crack. Let us represent the state of strain in an examined plate (Fig. 41) in the form of the sum of the state of strain in a solid plate (without crack), when in infinitely remote points stresses[3] p and q act, and the state of strain in a plate with a crack for $-l \leq x \leq l$, when on its sides are assigned the following boundary conditions:

and
$$-\sigma_y(x,0) = q_n = p\sin^2\alpha + q\cos^2\alpha,$$
$$-\tau_{xy}(x,0) = q_\tau = (p-q)\sin\alpha\cos\alpha, \tag{III.55}$$

and in points at infinity of the plate there are no stresses. Obviously, for an opening crack the following condition should hold:

$$p\sin^2\alpha + q\cos^2\alpha \geqslant 0. \tag{III.56}$$

The first state of strain does not depend on the dimensions of the crack and is characterized by limited components of the stress tensor all points of the plate. The second state of strain, in turn, can be represented as the sum of the state of strain in a plate with crack-cut ($-l \leq x \leq l$) considering the following boundary conditions on its sides:

$$1)\quad -\sigma_y(x,0) = q_n = p\sin^2\alpha + q\cos^2\alpha, \quad \tau_{xy}(x,0) = 0; \tag{III.57}$$

$$2)\quad \sigma_y(x,0) = 0, \quad -\tau_{xy}(x,0) = q_\tau = (p-q)\sin\alpha\cos\alpha. \tag{III.58}$$

Using the method of N. I. Muskhelishvili [103] (see Section 1 Chapter III), on the basis of boundary conditions (III.57) and (III.58) for determination of components of the stress tensor we will obtain the following complex potentials:

$$\Phi_1(z) = \frac{q_n l^2}{2\sqrt{z^2-l^2}\,(z+\sqrt{z^2-l^2})}, \quad \Psi_1(z) = \frac{q_n l^2 z}{2\sqrt{z^2-l^2}\,(z^2-l^2)}; \tag{III.59}$$

$$\Phi_2(z) = \frac{-iq_\tau l^2}{2\sqrt{z^2-l^2}\,(z+\sqrt{z^2-l^2})}, \tag{III.60}$$
$$\Psi_2(z) = \frac{-iq_\tau l^4\,(z+2\sqrt{z^2-l^2})}{2\,(z^2-l^2)^{\frac{3}{2}}\,(z+\sqrt{z^2-l^2})^2},$$

where functions (III.59) correspond to the problem with boundary conditions (III.57), and functions (III.60) to the problem with boundary conditions (III.58).

Let us introduce local system of polar coordinates r, β with origin in the tip of the crack and polar axis coinciding with the tangent to the sides of the crack in this point (Fig. 41). In this system of coordinates we will calculate the stress components σ_r, σ_β, $\tau_{r\beta}$ in the neighborhood of the ends of the examined crack, i.e., in points $z = \pm l + z_1$, where $z_1 = re^{i\beta}$, $r \ll l$. Using the formulas of Koloson-Muskhelishvili and functions (III.59) and (III.60) we will determine (just as is done in the preceding section) components $\sigma_r^{(2)}$, $\sigma_\beta^{(2)}$ and $\tau_{r\beta}^{(2)}$ for the second state of strain, and then will add to them corresponding components $\sigma_r^{(1)}$, $\sigma_\beta^{(1)}$ and $\tau_{r\beta}^{(1)}$ for the first state of strain. As a result we obtain

$$\left.\begin{aligned}
&\sigma_r = \frac{1}{4\sqrt{2r}}\left\{k_1\left(5\cos\frac{\beta}{2} - \cos\frac{3\beta}{2}\right) + \right.\\
&\left. + k_2\left(-5\sin\frac{\beta}{2} + 3\sin\frac{3\beta}{2}\right)\right\} + 0(1),\\
&\sigma_\beta = \frac{1}{4\sqrt{2r}}\left\{k_1\left(3\cos\frac{\beta}{2} + \cos\frac{3\beta}{2}\right) - 3k_2\left(\sin\frac{\beta}{2} + \sin\frac{3\beta}{2}\right)\right\} + 0(1),\\
&\tau_{r\beta} = \frac{1}{4\sqrt{2r}}\left\{k_1\left(\sin\frac{\beta}{2} + \sin\frac{3\beta}{2}\right) + \right.\\
&\left. + k_2\left(\cos\frac{\beta}{2} + 3\cos\frac{3\beta}{2}\right)\right\} + 0(1),
\end{aligned}\right\} \tag{III.61}$$

where

$$k_1 = p\sqrt{l}\,(\sin^2\alpha + \eta_0\cos^2\alpha), \quad k_2 = p\,(1 - \eta_0)\sqrt{l}\sin\alpha\cos\alpha; \tag{III.62}$$

0(1) - limited part of the component of the stress tensor when $r \to 0$; $\eta_0 = \frac{q}{p}$. Analogous expressions for components of the stress tensor in the neighborhood of the ends of a linear crack during flat extension of the plate are obtained by another means in work [219].

Using further formulas (III.61), equations (III.34) and (III.35) can be written as:

$$k_1^*\left(3\cos\frac{\beta_*}{2} + \cos\frac{3\beta_*}{2}\right) + 3k_2^*\left(\sin\frac{\beta_*}{2} + \sin\frac{3\beta_*}{2}\right) = \frac{4K\sqrt{2}}{\pi}, \tag{III.63}$$

$$k_1\left(\sin\frac{\beta_*}{2} + \sin\frac{3\beta_*}{2}\right) + k_2\left(\cos\frac{\beta_*}{2} + 3\cos\frac{3\beta_*}{2}\right) = 0, \tag{III.64}$$

where coefficients of intensity of stresses k_1^* and k_2^* are determined by formulas (III.62) when $p = p_{*,1}$ and $q = q_{*,1}$.

Thus, for the examined problem considering fulfillment of condition (III.56) on the basis of equation (III.63) we will obtain the following formula for determination of limit stresses $p_{*,1}$ and $q_{*,1}$:

$$p_{*,1} = \frac{K\sqrt{2}}{\pi\sqrt{l}} \frac{1}{\cos^2\frac{\beta_*}{2}\left\{\cos\frac{\beta_*}{2}(\sin^2\alpha+\eta_0\cos^2\alpha) - 3(1-\eta_0)\sin\alpha\cos\alpha\sin\frac{\beta_*}{2}\right\}} \quad (q_{*,1} = \eta_0 p_{*,1}), \tag{III.65}$$

where angle β_* is found from equation (III.64). Values of angle β_*, which provide the maximum value of intensity of elastic stresses σ_β are calculated by the formula

$$\beta_* = 2\,\mathrm{arctg}\,\frac{1 \mp \sqrt{1+8n^2}}{4n}, \tag{III.66}$$

where the "+" corresponds to values of $k_1 < 0$ and the "-" to values of $k_1 > 0$;

$$n = \frac{k_2}{k_1} = \frac{(1-\eta_0)\sin\alpha\cos\alpha}{\sin^2\alpha+\eta_0\cos^2\alpha}. \tag{III.67}$$

Formulas (III.65)-(III.67) were established in works [120, 133]. These formulas also solved the problem about limit stresses during extension-compression of an infinite plate weakened by a peaked narrow slot, if during the deformation of the plate the sides of the slot do not touch each other.

Analogous problems for a plate with an opened crack are examined in work [56, 138].

Closed crack. When a plate with a crack (Fig. 41) is subjected to extension-compression by external stresses p and q so that we have inequality

$$p \sin^2 \alpha + q \cos^2 \alpha < 0, \qquad \text{(III.68)}$$

then, obviously, during the deformation of the plate the sides of the crack will touch each other all along the length.

Analogously to the preceding case, the state of strain in a plate with a crack under condition (III.68) is represented in the form of the sum of the state of strain in a solid (without crack) infinite plate, subjected to deformation by external stresses p and q and the state of strain in a plate with a crack ($-l \leq x \leq l$) when on its sides the boundary conditions are assigned

$$\sigma_y(x, 0) = 0, \quad -\tau_{xy}(x, 0) = q_\tau^{(3)} = (p - q) \sin \alpha \cos \alpha - \\ - \rho_0 (p \sin^2 \alpha + q \cos^2 \alpha), \qquad \text{(III.69)}$$

where ρ_0 is the coefficient of sliding friction, which can appear between the sides of a crack at their contact [see (III.68)] and when they slide against one another; in points at infinity of the plate there are not stresses.

To calculate components of the stress tensor considering boundary conditions (III.69) we have complex potentials (III.60), where we should put $q_\tau = q_\tau^{(3)}$. Using the formulas of Kolosov-Muskhelishvili and functions (III.60) when $q_\tau = q_\tau^{(3)}$, and also considering the limitedness of components of stresses of the first state of strain, for our problem under condition (III.68) in the neighborhood of the ends of the crack components of the elastic stress tensor σ_r, σ_β, $\tau_{r\beta}$ in polar system of coordinates r, β are expressed by formulas (III.61) if in these formulas we set

$$k_1 = 0; \quad k_2 = \sqrt{l}\,[(p - q) \sin \alpha \cos \alpha - \rho_0 (p \sin^2 \alpha + q \cos^2 \alpha)]. \qquad \text{(III.70)}$$

On the basis of equations (III.34), (III.35) and expressions (III.61), (III.66), (III.67) and (III.70) to calculate limit stresses $p = p_{*,2}$ and $q = q_{*,2}$ considering fulfillment of condition (III.68) we have the formula

$$p_{*,2} = -\frac{\sqrt{2}K}{\pi\sqrt{l}} \frac{[(1-\eta_0)\sin\alpha\cos\alpha - \rho_0(\sin^2\alpha + \eta_0\cos^2\alpha)]^{-1}}{3\sin\frac{\beta_*}{2}\cos^2\frac{\beta_*}{2}}; \tag{III.71}$$

$$q_{*,2} = \eta_0 p_{*,2}; \quad \eta_0 = \frac{q}{p};$$
$$\beta_* = 2\,\mathrm{arctg}\frac{1}{\sqrt{2}} \qquad (\beta_* \approx 70^\circ 30'). \tag{III.72}$$

According to equality (III.72) formula (III.71) can be written as:

$$\begin{gathered} p_{*,2} = -\frac{\sqrt{2}K}{\pi\sqrt{l}} \cdot \frac{\sqrt{3}}{2}[(1-\eta_0)\sin\alpha\cos\alpha - \\ -\rho_0(\sin^2\alpha + \eta_0\cos^2\alpha)]^{-1}, \\ q_{*,2} = \eta_0 p_{*,2}. \end{gathered} \tag{III.73}$$

In the case of uniaxial compression, for example, when $p < 0$, $q = 0$, $\eta_0 = 0$, from formula (III.73) we find

$$-p_{*,2} = \frac{\sqrt{2}K}{\pi\sqrt{l}} \cdot \frac{\sqrt{3}}{2\sin\alpha}(\cos\alpha - \rho_0\sin\alpha)^{-1}. \tag{III.74}$$

Results of calculation by formulas (III.72) and (III.74) are easy to compare with experimental data.

Thus, for our problem the values of limit stresses are determined so: considering fulfillment of condition (III.56) $p_* = p_{*,1}$, $q_* = q_{*,1}$ and considering fulfillment of condition (III.68) $p_* = p_{*,2}$, $q_* = q_{*,2}$, where $p_{*,j}$ and $q_{*,j}$ are calculated by formulas (III.65) for $j = 1$ and formulas (III.73) for $j = 2$.

Diagram of limit stresses. Formulas (III.65) and (III.73) for determination of limit stresses during biaxial extension-compression of a plate with a crack (see Fig. 41) can yet be written as:

$$\begin{gathered} p_{*,j} = Rf_j(\alpha, \eta_0); \quad q_{*,j} = \eta_0 p_{*,j}, \\ t = p(\sin^2\alpha + \eta_0\cos^2\alpha), \end{gathered} \tag{III.75}$$

where when $t \geq 0$ we have $j = 1$ and when $t < 0$ we have $j = 2$.

In formula (III.75)

$$R = \frac{K\sqrt{2}}{\pi\sqrt{l}}; \qquad \text{(III.76)}$$

$$f_1(\alpha, \eta_0) = \frac{1}{\cos^3 \frac{\beta_*}{2}} \left[\cos \frac{\beta_*}{2} (\sin^2 \alpha + \eta_0 \cos^2 \alpha) - \right.$$

$$\left. - \frac{3}{2} (1 - \eta_0) \sin 2\alpha \sin \frac{\beta_*}{2}\right]^{-1}; \qquad \text{(III.77)}$$

$$f_2(\alpha, \eta_0) = -\frac{\sqrt{3}}{2} [(1 - \eta_0) \sin \alpha \cos \alpha -$$

$$- \rho_0 (\sin^2 \alpha + \eta_0 \cos^2 \alpha)]^{-1}, \qquad \text{(III.78)}$$

where $\eta_0 = \frac{q}{p}$; angle β_* is determined by formulas (III.65) and (III.66).

If value of parameters l, η_0, ρ_0, and α are known, then, using the above formulas, one can determine stresses p_* and q_* in every concrete case.

Constructing on the basis of formulas (III.75)-(III.78) the diagrams of limit stresses during plane stress or plane deformation of a body has an important value. In plane pOq they constitute the boundary of the region of change of stresses p and q, permissable with respect to the propagation of arbitrarily oriented linear cracks of assigned length in a deformable solid.

These diagrams are useful to evaluate the danger of rupture of deformable solids, when the usual phenomenological hypotheses cannot be used [161]. This pertains mainly to brittle rupture of solids whose structure contains fracture-type defects. In such cases brittle rupture, as it is known, is connected with the appearance of conditions with which fracture-type defects go into a state of maximum equilibrium and, consequently, with small disturbances of the field of external stresses the possibility of their propagation across the cross section of the solid develops.

Let us construct the diagram of limit stresses during the plane stress of a brittle body weakened by defects of the linear crack type. Let us assume that in the initial (undeformed) brittle body are internal defects of fracture type, whose characteristic linear

dimension is $2l$, and let us assume that such defects are randomly oriented and are dispersed all over the volume of the body, so that they can be taken isolated from each other. If such a body is subjected to uniaxial extension by external stresses p, then, taking the characteristic linear dimension of the defect as the length of the crack and using formulas (III.75)-(III.77), when $\eta_0 = 0$ we find that the minimum value of tensile stresses is expressed by the equality

$$p_*^{(\min)} = Rf(\alpha_*, 0) \approx 0{,}97R, \quad q_*^{(\min)} = 0.$$

Here α_* is the value of angle α at which function $f(\alpha, 0)$, represented by equality (III.77), has a minimum. For our case this angle is 1.19 rad (Fig. 39). Inasmuch as the body contains defects (cracks) of different orientation, their number must always include cracks for which $\alpha = \alpha_*$.

In accordance with formulas (III.76) for a given material under preassigned conditions (temperature, environment, character of heterogeneity of structure and others) R is constant.

Under the examined conditions $p_*^{\min}$ is the mean value of technical strength of a given material during uniaxial extension:

$$p_*^{(\min)} = 0{,}97R \approx \sigma_b,$$

where σ_b is the technical strength of a brittle material under uniaxial extension.

Hence

$$R \approx 1{,}03\sigma_b.$$

If a body with cracks is subjected to biaxial extension-compression by stresses p and $q(\eta_0 \neq 0)$, the value of angle $\alpha = \alpha_*$, at which limit stresses p_* and q_* have a minimum, is unknown beforehand.

But also in this case, in the number of all possible oriented cracks of average length $2l$ such a direction should exist. It is determined by certain angle $0 < \alpha_* \leq \frac{\pi}{2}$, at which the minimum value is taken (at a given value of η_0) either by function $f_1(\alpha, \eta_0)$ when $t \geq 0$, or function $f_2(\alpha, \eta_0)$ when $t < 0$.

Let us consider the case when $t \geq 0$. Then angle α_* must satisfy the conditions

$$f_1'(\alpha_*, \eta_0) = 0, \quad f_1''(\alpha_*, \eta_0) > 0, \tag{III.79}$$

where the primes designate differentiation with respect to α.

If angle α_* is determined, then by formulas (III.75)-(III.78) one can determine limit stresses $p_*^{(min)}$ and $q_*^{(min)}$.

Determination of angles α_* from equations (III.79) in general is difficult. On Fig. 42 are built graphs of the change of function $f_1(\alpha, \eta_0)$ when $0 < \alpha \leq \frac{\pi}{2}$. Using these graphs, one can determine $p_{*,1}^{(min)}$, $q_{*,1}^{(min)}$ for values of η_0 shown on Fig. 42, and then by formulas (III.75) find $\frac{p_{*,1}^{(min)}}{R}$ and $\frac{q_{*,1}^{(min)}}{R}$.

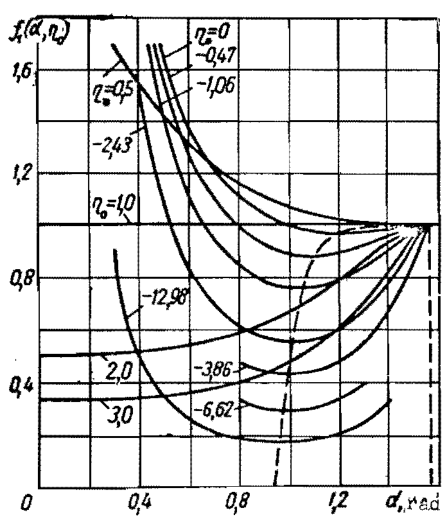


Fig. 42.

In such a way Table 12 is composed, according to which on Fig. 43a the change of limit stresses as a function of parameter η_0 is graphed, where along the axis of abscissas are plotted values of $\frac{p_*^{(min)}}{\sigma_b}$, and along the axis of ordinates the values of

$$\frac{q_*^{(min)}}{\sigma_b} \quad (\sigma_b = 0{,}97R).$$

Curve 1 on Fig. 43 is the diagram of limit stresses $p_*^{(min)}$ and $q_*^{(min)}$ when $t \geq 0$.

Table 12

η_0	$\frac{p_{*,1}^{(min)}}{R}$	$\frac{q_{*,1}^{(min)}}{R}$	α_*	β_*	η_0	$\frac{p_{*1}^{(min)}}{R}\left(\frac{q_{*1}^{(min)}}{R}=1;\ \alpha_*=0;\ \beta_*=0\right)$
$-\infty$	0,00	−2,58	53° 38′	−84° 54′	1,00	1,00
−12,98	0,17	−2,17	54° 49′	−79° 58′	1,20	0,83
−6,616	0,29	−1,94	55° 38′	−77° 20′	1,50	0,67
−3,860	0,43	−1,65	56° 57′	−74° 08′	2,00	0,50
−2,43	0,55	−1,35	57° 59′	−70° 28′	3,00	0,33
−1,06	0,75	−0,80	59° 59′	−62° 40′	4,00	0,25
−0,47	0,88	−0,41	62° 17′	−50° 30′	6,00	0,17
0,00	0,97	0,00	68° 18′	−34° 43′	8,00	0,13
0,50	1,00	0,50	90°	0°	10,00	0,10

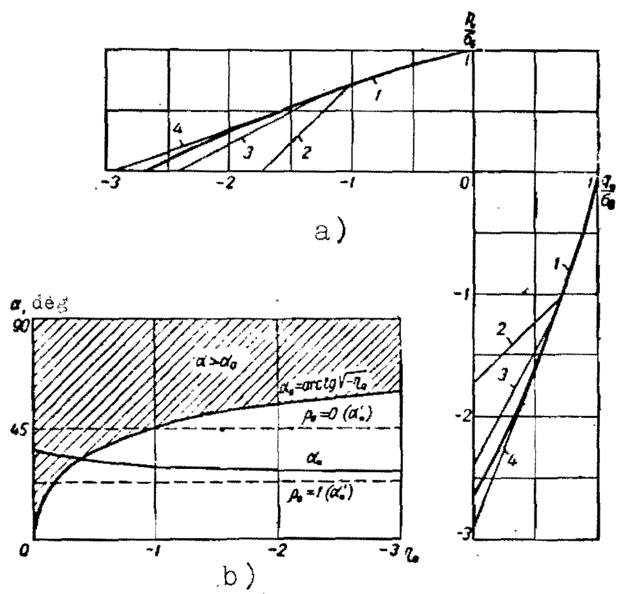


Fig. 43.

This diagram is valid for all slot-cracks, whose sides do not touch each other during deformation of a body. Moreover $\frac{p_*^{(min)}}{\sigma_b}$ and $\frac{q_*^{(min)}}{\sigma_b}$ should be examined as relative value of the main stresses appearing in a deformable body during plane stress.

We will construct diagrams of limit stresses for $t < 0$. Let us assume that $p < 0$, $q > 0 (0 \geq \eta_0 \geq -\infty)$. It is obvious that the diagram of limit stresses when $p < 0$, $q > 0$ and $p > 0$, $q < 0$ are symmetric with respect to the bisector of the first and third quadrants of plane pOq.

Just as before, we minimize function $f_2(\alpha, \eta_0)$, represented by formula (III.78), by angle α, when $0 \leq \alpha \leq \frac{\pi}{2}$ and

$$p(\sin^2\alpha + \eta_0 \cos^2\alpha) < 0, \quad p > 0, \quad 0 \geqslant \eta_0 \geqslant -\infty. \qquad \text{(III.80)}$$

In this case we will find that function $f_2(\alpha, \eta_0)$ takes minimum value when $\alpha = \alpha_*'$, where

$$\alpha_*' = \frac{1}{2} \operatorname{arctg} \frac{1}{\rho_*}. \qquad \text{(III.81)}$$

On the basis of equality (III.81) and formulas (III.75) we have

$$p_{*,2}^{(min)} = \frac{-R\sqrt{3}}{(1-\eta_0)\sqrt{1+\rho_0^2} - \rho_0(1+\eta_0)}; \quad q_{*,2}^{(min)} = \eta_0 p_{*,2}^{(min)}. \qquad \text{(III.82)}$$

Condition (III.80) when $p < 0$ can be written as:

$$\alpha > \alpha_0; \quad p < 0, \quad 0 \geqslant \eta_0 \geqslant -\infty, \qquad \text{(III.83)}$$

where $\alpha_0 = \operatorname{arctg} \sqrt{-\eta_0}$.

Formulas (III.83) can be used only when for given values of parameters η_0 and ρ_0 angle $\alpha = \alpha_*'$ satisfied inequality (III.83).

At certain values of parameters η_0 and ρ_0 condition (III.83) does not hold for angles $\alpha = \alpha_*^!$. This means that cracks oriented at an angle $\alpha_*^!$ for given values of η_0 and ρ_0 do not close and, consequently, for such cracks limit stresses cannot be calculated by formulas (III.82). In this case local destruction of a body can be predetermined by the propagation of the opening crack ($t \geq 0$).

In the general case of the plane stressed-deformed state of a brittle body with cracks, i.e., when $0 \leq \rho_0 \leq 1$, $0 \geq \eta_0 \geq -\infty$, limit stresses $p_*^{(min)}$ and $q_*^{(min)}$ will be least values of stresses, $p_{*,j}$ and $q_{*,j}$ calculated by formula (III.75) both when $t \geq 0$ and $t < 0$. Having this in mind on Fig. 43b the change of angles α_0 and α_* as a function of η_0 [angles α_* satisfy condition (III.79)] is graphed. The broken lines on the figure show the change of angles $\alpha_*^!$ for $\rho_0 = 0$ and $\rho_0 = 1$.

Using graphs on Fig. 43b and formulas (III.75) for our case ($p < 0$, $q > 0$, $0 \geq \eta_0 \geq -\infty$), we obtain the following formula:

I. If $\alpha_*^! > \alpha_0$ and $\alpha_* > \alpha_0$, then

$$p_*^{(min)} = \min\{Rf_1(\alpha_0, \eta_0);\quad p_{*,1}^{(min)}\};\quad q_*^{(min)} = \eta_0 p_*^{(min)}, \tag{III.84}$$

where

$$\alpha_0 = \mathrm{arctg}\sqrt{-\eta_0}.$$

II. If $\alpha_*^! < \alpha_0$ and $\alpha_* > \alpha_0$, then

$$p_*^{(min)} = \min\{Rf_1(\alpha_0, \eta_0);\quad Rf_2(\alpha_0, \eta_0)\},\quad q_*^{(min)} = \eta_0 p_*^{(min)}. \tag{III.85}$$

III. If $\alpha_*^! < \alpha_0$ and $\alpha_* \leq \alpha_0$, then

$$p_*^{(min)} = \min\{p_{*,1}^{(min)};\quad Rf_2(\alpha_0, \eta_0)\},\quad q_*^{(min)} = \eta_0 p_*^{(min)}. \tag{III.86}$$

IV. If $\alpha_*^! > \alpha_0$ and $\alpha_* \leq \alpha_0$, then

$$p_*^{(\min)} = \min\{p_{*,1}^{(\min)};\ \ p_{*,2}^{(\min)}\},\quad q_*^{(\min)} = \eta_0 p_*^{(\min)}, \qquad \text{(III.87)}$$

where functions $f_1(\alpha, \eta_0)$ and $f_2(\alpha, \eta_0)$ are represented by equalities (III.77) and (III.78); the value of $p_{*,2}^{(\min)}$ is determined by formula (III.82), and $p_{*,1}^{(\min)}$ is calculated on the basis of formula (III.75) (Table 12). For our case formulas (III.84)-(III.87) are used on Fig. 43a to build the diagram of limit stresses $p_*^{(\min)}$ and $q_*^{(\min)}$ when $\rho_0 = 0$ (curve 2), $\rho_0 = 0.3$ (curve 3) and $\rho_0 = 0.5$ (curve 4).

A positive feature of diagrams of limit stresses is that they are obtained as a result of analysis of the development of defects inherent to a material – cracks – and, consequently, they reflect one of the physical processes of destruction.

In conclusion let us note that in work [100] an attempt also is undertaken to construct a diagram of limit stresses, proceeding from the power principle of Griffith and the hypothesis fact that propagation of an initial crack always (at any direction of the effective loads) occurs in the plane of location of the crack. However, this hypothesis is not confirmed by experimental data.

6. Strength Criterion During Biaxial Extension

The selection of a strength criterion in the brittle rupture of materials in a biaxial state of strain has not yet been finally resolved. The most acceptable criteria are the hypothesis of the greatest normal (stretching) stresses [46] and the theory of Griffith [191].

The Griffith theory about brittle rupture in the biaxial state of strain, as it is known, is based on analysis of elastic stresses near a stretched elliptic strip in a plate subjected biaxial extension. It is assumed that the radius of curvature of such a space in its tip, although it is small, has a finite value[4] (for material concerning this question see work [204, 207, 209, 210]).

Physical prerequisites of the theory are as follows. It is accepted that real material in all directions contains elliptic cavities — cracks of identical length. Destruction of such a body sets in when the greatest local tensile stress for the most dangerous orientation of a cavity (cracks) reaches σ_B.

Determining maximum tensile stresses on the contour of an elliptic hole in an elastic plate subjected to biaxial extension by external stresses p and q considering the most unprofitable location of a space in the file of effective stresses and using the above consideration, Griffith [191] formulated the following criterion of brittle rupture of a solid under plane stress: 1) if $3p + q > 0$, rupture occurs at $p = \sigma_B$; 2) if $3p + q < 0$, rupture sets in when

$$(p-q)^2 + 8\sigma_B(p+q) = 0.$$

Graphic interpretation of these equations is given on Fig. 44 in the form of line 2. The criterion of brittle strength according to the hypothesis of the biggest normal stresses is rather simple $p = \sigma_B$ or $q = \sigma_B$ (line 3). On this figure line 1 is the diagram of limit stresses built according to equations (III.65) (see curve 1 on Fig. 43).

Let us compare diagrams on Fig. 44 with experimental data obtained during a test of brittle bodies. In the beginning we will examine certain experimental results obtained during the test (destruction) of tubular cast-iron samples under plane stress[5] [181, 182, 188].

Cast iron is a material with rather limited plasticity and with a particular heat treatment in its structure there are numerous (randomly oriented in the volume of the body) graphite inclusions in the form of thin platelets. Graphite platelets have minute tensile strength as compared to the ferrite base of cast iron, consequently, they can be examined in the first approximation as defects of the peaked cavity-type cracks in the basic structure of a material.

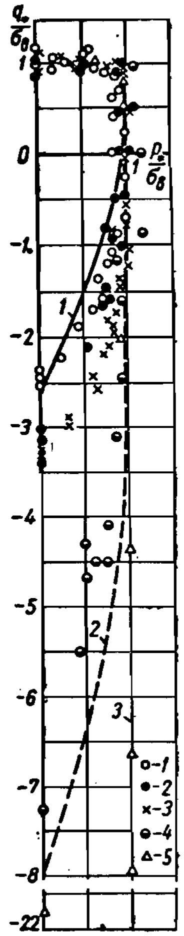


Fig. 44.

Thus, cast iron with laminar graphite in the first approximation is a suitable real object of the above reference model of a brittle body with cracks. Because of this, results of tests on cast-iron samples (whose material has the shown structure) during plane stress are interesting to compare with theoretical data.

On Fig. 44 in coordinates $\frac{q}{\sigma_B}$ and $\frac{p}{\sigma_B}$ are given results of experiments [181, 182, 188] on the rupture of cast-iron tubular samples under plane stress. Points 1 pertain to samples from modified cast iron (σ_B = 345.3 N/mm^2), and points 2 and 3 to samples from gray cast iron (correspondingly, σ_B = 185.4 and σ_B = 228.6 N/mm^2).

As can be seen from Fig. 44, when $p > 0$ and $q > 0$ experimental data will agree well with results of theory, but when $q < 0$, $p > 0$ they lie somewhat lower than curve 1.

Certain strength characteristics of the cast iron (technical strength during uniaxial extension σ_B and the relation of technical strength during compression $\sigma_Ж$ to technical tensile strength — coefficient $k_0 = \frac{\sigma_Ж}{\sigma_B}$) and the average linear dimension of graphite inclusions l are given in Table 13. For these materials the experimental value of coefficient k_0 changes within limits of 2.5-3.3, the theoretical value, calculated by formula (III.65), is 2.7, and calculated on the basis of the theory of Griffith it is 8.

Table 13.

Material	σ_B, N/mm^2	k_0	l, мм
Gray cast iron........	185,4	3,2	0,64
Gray cast iron........	228,6	3,3	0,57
Inoculated cast iron...........	345,3	2,5	0,38

When the sides of an initial (responsible for destruction) crack touch each other, coefficient k_0 depends on the value of coefficient of friction ρ_0 (see diagram on Fig. 43); when $\rho_0 > 0$ we have $k_0 > 1.7$.

Points 4 and 5 on Fig. 44 are results obtained by Davidenkov and Stavrogin [46] during tests on the rupture of gypsum and glass tublar samples under plane stress. Obtained data in the first quadrant ($q > 0$, $p > 0$) will agree well with the diagram of limit stresses; in the second quadrant ($q < 0$, $p > 0$) they deviate from this diagram. The strength of the glass samples in the second quadrant is better described by line 3 (although besides is not clear mechanism of destruction under conditions, close to pure compression), and the strength of gypsum is in the region between lines 1 and 2. According to data of Davidenkov and Stavrogin, for gypsum samples $\sigma_B = 4.12$ daN/mm^2, $k_0 = 7.5$; for glass $\sigma_B \approx 3.92$ daN/mm^2, $k_0 = 22$.

Thus, theoretically established (on the basis of a model of an ideally brittle body with cracks) general regularities of the rupture of brittle materials with clearly expressed defectiveness of structure

in the form of narrow slot-cracks (as occurs for example in the first approximation for cast iron with laminar graphite inclusions) the diagram of limit stresses (curve 1 on Fig. 44) will agree satisfactorily with results of experiments quantitatively if we consider that it characterizes stresses p_* and q_*, necessary for local breakdown of crack propagation in a brittle body. However, local crack propagation does not always lead to complete rupture of a body. Complete rupture of a body happens when after external stresses reach their limit levels there occurs unstable (spontaneous) crack propagation (see note on p. **xvii**). Under compression stresses local crack propagation in general is stable, for which the destruction of a body with fracture defects decreasing compression requires stresses $p > p_*$ and $q > q_*$. Consequently, in such a case, the rupture stress will be characterized in plane pOq by a point located beyond the diagram of limit stresses (see Fig. 44).

The theory of crack propagation in a deformable brittle solid also makes it possible to predict a considerable excess of strength of a material during compression over tensile strength. This phenomenon is observed also during a test of brittle bodies.

Probabily, for brittle bodies in whose structure there is no sufficiently well-developed fracture defects, the model of a brittle body with linear (peaked or rounded at the tip) cracks does not reflect the behavior of such bodies under conditions of plane stress with great compression stresses completely enough, (see, for example, point 5 on Fig. 44).

In such cases defects of other configuration can play an essential role in the rupture of brittle materials namely, linear cracks, peaked cavities, accumulations of embryonic linear cracks, etc. For this reason a more full description of the behavior of brittle bodies under load, especially in a region of great compression stresses and the construction of corresponding criteria of brittle strength require the study of regularities of the development of defects of different structure during the deformation of a solid,

clarification of their mutual influence and then construction of diagrams of limit stresses.[6]

At the same time, already on the basis of comparisons of the results of theory and experiment conducted in this section, it may be concluded that the examined theoretical diagram well describes the behavior of different brittle bodies under conditions of plane stress, if main stresses $p > 0$ and $q > 0$. In this case the position that the rupture of brittle bodies is controlled by the development of fracture defects in the structure of the deformable material is confirmed. Characteristically, in the shown region of stresses ($p > 0$, $q > 0$) the obtained diagrams of limit stresses (see Fig. 44) practically confirm the hypothesis of greatest normal (stretching) stresses, known in engineering practice.

7. Extension of a Plate with Curved Crack

Investigation of the maximum equilibrium of an infinite uniform plane with cracks (cuts) along the arcs of the circumference is the subject of work [13, 131, 134, 136]. In these works on the basis of equations (III.34) and (III.35) formulas are set up for figuring the limit loads in the case of one [131], two [19] and a system [136] of curved cracks.

Below a solution is given to the problem about maximum equilibrium of an infinite plane with one curved crack and certain peculiarities of the propagation of curved cracks in plates are set up.

Let us consider a crack in the form of a cut along the arc of the circumference. Let us assume that an infinite plate of unit thickness is weakened by a curved crack (cut along arc of circumference with radius R), so that 2Θ is the central angle of the arc of the cut-crack. Let us refer the plate to rectangular cartesian coordinates coincides with the center of the arc of the cut, and plane xOy lies on the middle plane of the plate; axis Ox is directed to the middle of the crack. Further we will assume that the edges of the

crack are free from external stresses, and in points at infinity of the plate uniformly distributed stresses p and q act in mutually perpendicular directions; besides stresses are directed at an angle α to axis Ox (Fig. 45).

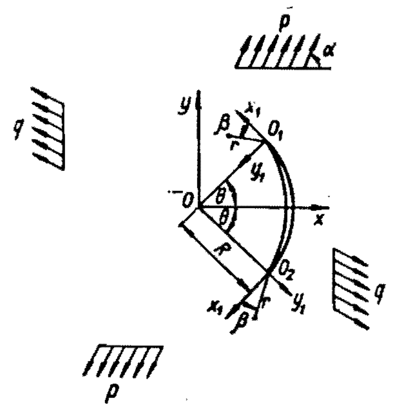


Fig. 45.

The problem is the determination of limit values of external stresses $p = p_*$ and $q = q_*$. To solve the problem first we find the distribution of stresses in the environment of the ends of the examined crack. Let us note that Muskhelishvili $\Phi(\zeta)$ and $\Omega(\zeta)$ functions determining the stressed-deformed state of a plate with a cut along the arc of circumference of radius R, have the following form [103]:

$$\Phi(\zeta) = \frac{1}{2X(\zeta)}\left\{c_0\zeta + c_1 + \frac{D_1}{\zeta} + \frac{D_2}{\zeta^2}\right\} + \frac{D_0}{2} + \frac{\overline{\Gamma'}}{2\zeta^2}, \qquad \text{(III.88)}$$

$$\Omega(\zeta) = \frac{1}{2X(\zeta)}\left\{c_0\zeta + c_1 + \frac{D_1}{\zeta} + \frac{D_2}{\zeta^2}\right\} - \frac{D_0}{2} - \frac{\overline{\Gamma'}}{2\zeta^2}. \qquad \text{(III.89)}$$

where

$$\zeta = \frac{z}{R}; \quad X(\zeta) = \sqrt{(\zeta - e^{i\theta})(\zeta - e^{-i\theta})} = \sqrt{\zeta^2 - 2\zeta\cos\Theta + 1}.$$

Here by $X(\zeta)$ is implied the branch for which $\lim \zeta^{-1}X(\zeta) = 1$ as $\zeta \to \infty$. In this case we have $X(0) = -1$. Hence it is easy to establish the following equality:

$$X\left(\frac{1}{\zeta}\right) = -\frac{1}{\zeta}X(\zeta). \qquad \text{(III.90)}$$

Constants c_0, c_1, D_0, D_1, D_2 for the problem are determined by the relationships

$$c_0 = \frac{1}{2}(\Gamma' - \overline{\Gamma'})\sin^2\frac{\Theta}{2} + \frac{4\Gamma + (\Gamma' + \overline{\Gamma'})\sin^2\frac{\Theta}{2}\cos^2\frac{\Theta}{2}}{2\left(1 + \sin^2\frac{\Theta}{2}\right)};$$
$$c_1 = -c_0\cos\Theta, \quad D_0 = 2\Gamma - c_0;$$
$$D_1 = -\overline{\Gamma'}\cos\Theta, \quad D_2 = \overline{\Gamma'}, \tag{III.91}$$

where Γ and Γ' are connected with the main stresses on infinity by the formulas

$$\Gamma = \frac{1}{4}(p+q); \quad \Gamma' = -\frac{1}{2}(p-q)e^{-2i\alpha}. \tag{III.92}$$

The analytic formula of $\Psi(\zeta)$, which is encountered in expressions for determination of the component of the stresses tensor, is connected with functions $\Phi(\zeta)$ and $\Omega(\zeta)$ in the following way:

$$\Psi(\zeta) = \frac{1}{\zeta^2}\Phi(\zeta) - \frac{1}{\zeta^2}\overline{\Omega}\left(\frac{1}{\zeta}\right) - \frac{1}{\zeta}\Phi'(\zeta). \tag{III.93}$$

On the basis of formulas (III.88), (III.89) and (III.93) it is easy to establish the expression

$$\Psi(z) = \frac{1}{2\sqrt{z^2 - 2zR\cos\Theta + R^2}}\left\{\overline{D}_2 z + \overline{D}_1 R + \frac{c_0 + \overline{c}_1}{z}R^2 + \right.$$
$$+ \frac{\overline{c}_0 + c_1}{z^2}R^3 + \frac{D_1}{z^3}R^4 + \frac{D_2}{z^4}R^5 - \frac{1}{z^2 - 2zR\cos\Theta + R^2}\times$$
$$\times\left[\frac{(c_0 + c_1\cos\Theta - 2D_1)}{z}R^4 + \frac{3(D_1\cos\Theta - D_2)}{z^2}R^5 + \right.$$
$$\left.\left. + \frac{5D_2\cos\Theta - D_1}{z^3}R^6 - \frac{2D_2}{z^4}R^7\right]\right\} + \frac{\Gamma'}{2} + \frac{\operatorname{Re} D_0}{z^2}R^2 + \frac{3\overline{\Gamma'}}{2z^4}R^4$$
$$(z = R\zeta). \tag{III.94}$$

According to expression (III.88), we have

$$\Phi(z) = \frac{1}{2\sqrt{z^2 - 2zR\cos\Theta + R^2}}\left\{c_0 z + c_1 R + \frac{D_1}{z}R^2 + \frac{D^2}{z^2}R^3\right\} +$$
$$+ \frac{D_0}{2} + \frac{\overline{\Gamma'}}{2z^2}R^2. \tag{III.95}$$

Considering study of the state of strain in the environment of the end of a crack, let us turn to new coordinates with origin at the tip. Axis O_jx_1 $(j = 1, 2)$, corresponding to axis Ox in system xOy, is directed along the tangent to the surface of the crack at end O_1 or O_2 (see Fig. 45).

Functions $\Phi_1(z_1)$ and $\Psi_1(z_1)(z_1 = x_1 + iy_1 = re^{i\beta})$, playing the same role in the new system of coordinates $x_1O_jy_1$ as functions $\Phi(z)$ and $\Psi(z)$ in system of coordinates xOy, are determined by the following transition formulas:

$$\Phi_1(z_1) = \Phi(z_1e^{i\omega} + z_0);$$
$$\Psi_1(z_1) = [\Psi(z_1e^{i\omega} + z_0) + \bar{z}_0\Phi'(z_1e^{i\omega} + z_0)]\,e^{2i\omega}. \qquad \text{(III.96)}$$

When the origin of the new coordinates is at the tip of the crack O_1, in formulas (III.96) one should take $z_0 = Re^{i\theta}$, $\omega = \frac{\pi}{2} + \theta$. If however, the origin of the new coordinates is placed at the tip of the crack O_2, then $z_0 = Re^{-i\theta}$, and $\omega = -(\frac{\pi}{2} + \theta)$.

The components of the stress tensor σ_r, σ_β, $\tau_{r\beta}$ in the polar coordinates (r, β) (see Fig. 45) are determined by formulas [103]

$$\sigma_r + \sigma_\beta = 4\,\mathrm{Re}\,\Phi_1(z_1);$$
$$\sigma_\beta - i\tau_{r\beta} = 2\,\mathrm{Re}\,\Phi_1(z_1) + \overline{z_1\Phi_1'(z_1)} + \frac{\bar{z}_1}{z_1}\overline{\Psi_1(z_1)}. \qquad \text{(III.97)}$$

Inasmuch as we are interested in the distribution of stresses in the environment of the ends of the crack, i.e., in points $z_1 = re^{i\beta}$ when $r << R$ and $0 < \theta < \pi$, then subsequently we use the following decomposition: when $z = z_1e^{i\omega} + z_0 = Re^{i\theta} + ire^{i(\beta + \theta)}$ we have

$$z^n = R^ne^{n\theta i}\left(1 + \frac{r}{R}ie^{i\beta}\right)^n = R^ne^{n\theta i}\left[1 + n\frac{r}{R}ie^{i\beta} + 0\left(\frac{r^2}{R^2}\right)\right];$$
$$\frac{1}{\sqrt{(z - Re^{i\theta})(z - Re^{-i\theta})}} = -\frac{ie^{-\frac{\beta+\theta}{2}i}}{\sqrt{2Rr\sin\theta}}\left[1 - \frac{r}{R}\cdot\frac{e^{-i(\beta+\theta)}}{4\sin\theta} + 0\left(\frac{r^2}{R^2}\right)\right]; \qquad \text{(III.98)}$$

when $z = Re^{-i\theta} - ire^{(\beta - \theta)i}$ we obtain

$$z^n = R^n e^{-n\theta i}\left(1 - \frac{r}{R} ie^{i\beta}\right)^n =$$
$$= R^n e^{-n\theta i}\left[1 - n\frac{r}{R} ie^{i\beta} + 0\left(\frac{r^2}{R^2}\right)\right];$$
$$\frac{1}{\sqrt{(z - Re^{i\theta})(z - Re^{-i\theta})}} = \frac{ie^{-\frac{\beta-\theta}{2}i}}{\sqrt{2Rr\sin\theta}}\left[1 - \frac{r}{R} \times\right.$$
$$\left.\times \frac{e^{(\beta-\theta)i}}{4\sin\theta} + 0\left(\frac{r^2}{R^2}\right)\right].$$

Using relationships (III.98), on the basis of formulas (III.94)-(III.97) after necessary transformations we find the following expression for components of the stress tensor near the ends O_j ($j = 1, 2$) of a curved crack:

$$\sigma_r = \frac{1}{4\sqrt{2r}}\left\{k_{1,j}\left[5\cos\frac{\beta}{2} - \cos\frac{3}{2}\beta\right] + k_{2,j}\left[-5\sin\frac{\beta}{2} + \right.\right.$$
$$\left.\left. + 3\sin\frac{3}{2}\beta\right]\right\} + 4A_j\cos^2\beta + 0(r^{1/2}); \qquad \text{(III.99)}$$
$$\sigma_\beta = \frac{1}{4\sqrt{2r}}\left\{k_{1,j}\left[3\cos\frac{\beta}{2} + \cos\frac{3}{2}\beta\right] - 3k_{2,j}\left[\sin\frac{\beta}{2} + \sin\frac{3}{2}\beta\right]\right\} +$$
$$+ 4A_j\sin^2\beta + 0(r^{1/2});$$
$$\tau_{r\beta} = \frac{1}{4\sqrt{2r}}\left\{k_{1,j}\left[\sin\frac{\beta}{2} + \sin\frac{3}{2}\beta\right] + k_{2,j}\left[\cos\frac{\beta}{2} + 3\cos\frac{3}{2}\beta\right]\right\} -$$
$$- 2A_j\sin 2\beta + 0(r^{1/2}).$$

Here coefficients $k_{1,j}$ and $k_{2,j}$ ($j = 1, 2$) are determined by such formulas: in the environment of tip O_1 of a crack we have

$$k_{1,1} = \sqrt{R\sin\theta}\,\varphi_1(\alpha, \theta); \quad k_{2,1} = \sqrt{R\sin\theta}\,\varphi_2(\alpha, \theta);$$
$$A_1(\alpha, \theta) = \frac{1}{4}(p + q)\frac{\sin^2\frac{\theta}{2}}{1 + \sin^2\frac{\theta}{2}} + \qquad \text{(III.100)}$$
$$+ \frac{p - q}{4}\left[\frac{\cos 2\alpha\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{1 + \sin^2\frac{\theta}{2}} - \cos 2(\alpha - \theta)\right],$$

where

$$\varphi_1(\alpha, \Theta) = \frac{1}{2}\left[\frac{p + q - (p - q)\cos 2\alpha \sin^2\frac{\Theta}{2}\cos^2\frac{\Theta}{2}}{1 + \sin^2\frac{\Theta}{2}}\cos\frac{\Theta}{2} + \right.$$
$$\left. + (p - q)\sin 2\alpha \sin^3\frac{\Theta}{2} + (p - q)\cos\left(2\alpha - \frac{3}{2}\Theta\right)\right]; \qquad \text{(III.101)}$$
$$\varphi_2(\alpha, \Theta) = \frac{1}{2}\left[\frac{p + q - (p - q)\cos 2\alpha \sin^2\frac{\Theta}{2}\cos^2\frac{\Theta}{2}}{1 + \sin^2\frac{\Theta}{2}}\sin\frac{\Theta}{2} - \right.$$
$$\left. - (p - q)\sin 2\alpha \sin^2\frac{\Theta}{2}\cos\frac{\Theta}{2} - (p - q)\sin\left(2\alpha - \frac{3}{2}\Theta\right)\right] \quad (p \geqslant q),$$

in the environment of tip O_2 of a crack we have the relationships

$$k_{1,2} = \sqrt{R\sin\Theta}\,\varphi_1(\alpha, -\Theta); \quad k_{2,2} = \sqrt{R\sin\Theta}\,\varphi_2(\alpha, -\Theta); \qquad \text{(III.102)}$$
$$A_2(\alpha, \Theta) = A_1(\alpha, -\Theta).$$

If we ignore concrete values of k_{1j}, k_{2j}, A_j, then it is clear that expressions of (III.99) coincide with analogous expressions for a linear crack [see (III.61)]. Therefore it may be concluded that stresses in the environment of the tip of a linear crack during plane extension are determined by expressions (III.99) in the general load of the plate. Qualitatively this is confirmed also by experimental investigations [215].

For our problem (see Fig. 45) coefficients $k_{1,j}$ and $k_{2,j}$ are found directly from consideration of the distribution of stresses near the tip of the curved crack. According to formulas (III.97) and (III.99), these coefficients can be determined also from the relationship

$$\sigma_r + \sigma_\beta = k_{1,1}\sqrt{\frac{2}{r}}\cos\frac{\beta}{2} - k_{2,1}\sqrt{\frac{2}{r}}\sin\frac{\beta}{2} + 4A_1 + \qquad \text{(III.103)}$$
$$+ 0(r^{1/2}) = 4\operatorname{Re}\Phi_1(z_1).$$

By using decomposition (III.98), the right side of equality (III.103) can be represented in a form analogous to its left part. Then by simple comparison it is easy to determine during identical

harmonics the values of coefficients $k_{1,1}$, $k_{2,1}$ and A_1. Values of $k_{1,2}$, $k_{2,2}$ and A_2 are obtained automatically on the basis of relationships (III.102).

The idea of using relationship (III.103) for determination of coefficients of intensity of stresses was advanced in work [219]. In the same place expressions are given for coefficients $k_{1,1}$ and $k_{2,1}$ for certain cases of loading a plate with a crack. However in work [219] the expressions given for coefficients $k_{1,1}$ and $k_{2,1}$ for one end (O_1) of a curved crack in an infinite plate extended to infinity by stress p at an angle α to axis Ox (in our designations) are inaccurate. In this case, according to formulas (III.100) and (III.101), we have these expressions:

$$k_{1,1}=\frac{p\sqrt{R\sin\Theta}}{2}\left[\frac{1-\cos 2\alpha\sin^2\frac{\Theta}{2}\cos^2\frac{\Theta}{2}}{1+\sin^2\frac{\Theta}{2}}\cos\frac{\Theta}{2}+\right.$$
$$\left.+\sin 2\alpha\sin^3\frac{\Theta}{2}+\cos\left(2\alpha-\frac{3}{2}\Theta\right)\right];$$
$$k_{2,1}=\frac{p\sqrt{R\sin\Theta}}{2}\left[\frac{1-\cos 2\alpha\sin^2\frac{\Theta}{2}\cos^2\frac{\Theta}{2}}{1+\sin^2\frac{\Theta}{2}}\sin\frac{\Theta}{2}-\right.$$
$$\left.-\sin 2\alpha\sin^2\frac{\Theta}{2}\cos\frac{\Theta}{2}-\sin\left(2\alpha-\frac{3}{2}\Theta\right)\right]. \qquad \text{(III.104)}$$

In the case omnidirectional extension, (p = q) we have

$$k_{1,1}=\frac{p\sqrt{R\sin\Theta}}{1+\sin^2\frac{\Theta}{2}}\cos\frac{\Theta}{2},$$
$$k_{2,1}=\frac{p\sqrt{R\sin\Theta}}{1+\sin^2\frac{\Theta}{2}}\sin\frac{\Theta}{2}. \qquad \text{(III.105)}$$

If values of $k_{1,j}$ and $k_{2,j}$ (j = 1, 2) are determined, then, using equations (III.34) and (III.35), it is easy to obtain formulas for determination of the limit stresses. Actually, for our problem, by substituting in these equations expressions for $\sigma_\beta(r, \beta)$ from formulas (III.99) and by carrying out necessary transformations, we find

$$k_{1,j}^{*}\left(3\cos\frac{\beta_*}{2}+\cos\frac{3}{2}\beta_*\right)-3k_{2,j}^{*}\left(\sin\frac{\beta_*}{2}+\right.$$
$$\left.+\sin\frac{3}{2}\beta_*\right)=\frac{4\sqrt{2K}}{\pi};$$
$$k_{1,j}\left(\sin\frac{\beta_*}{2}+\sin\frac{3}{2}\beta_*\right)+ \tag{III.106}$$
$$+k_{2,j}\left(\cos\frac{\beta_*}{2}+3\cos\frac{3}{2}\beta_*\right)=0 \qquad (j=1,2),$$

where coefficients $k_{1,j}$ and $k_{2,j}$ $(j = 1, 2)$ are determined by formulas (III.100)-(III.102), and $k_{1,j}^{*}$ and $k_{2,j}^{*}$ by the formulas, if we assume that $p = p_*$ and $q = q_*$.

Values of angle β_* which ensure a maximum value of the intensity of elastic stresses $\sigma_\beta(r, \beta)$, are determined by the formulas

$$\beta_* = \pm 2\arcsin\sqrt{\frac{6n_j^2+1-\sqrt{8n_j^2+1}}{2\left(9n_j^2+1\right)}} \quad \text{when} \quad k_{1,j}>0; \tag{III.107}$$
$$\beta_* = \pm 2\arcsin\sqrt{\frac{6n_j^2+1+\sqrt{8n_j^2+1}}{2\left(9n_j^2+1\right)}} \quad \text{when} \quad k_{1,j}<0,$$

where the "+" corresponds to values of $k_{2,j} < 0$ and "-" to values of $k_{2,j} > 0$; $n_j = \frac{k_{2,j}}{k_{1,j}}$ $(j = 1, 2)$.

Thus, by using these formulas and relationships (III.100)-(III.102) and (III.106), it is possible to calculate the limit stress for a plate weakened by an isolated curvilinear crack in the form of a cut along the arc of the circumference and subjected to biaxial extension by stresses p and q (see Fig. 45).

Let us consdier certain special cases of the problem.

1. An infinite plate weakened by an internal rectilinear crack of length $2l$ is extended by evenly distributed stresses q, applied in points at infinity of the plate at an angle α to the line of location of the crack. It is required to determine the limits stresses $q = q_*$.

To solve the problem we use formulas (III.100)-(III.102), and also equations (III.106) and (III.107). Considering in formulas (III.100)-(III.102) that $p = 0$, $q \neq 0$ and $\Theta \to 0$ and $R \to \infty$, but so that $R\Theta = l = \text{const}$, we find easily

$$k_{1,j} = q\sqrt{l}\sin^2\alpha,\ k_{2,j} = q\sqrt{l}\sin\alpha\cos\alpha \qquad (j = 1, 2);$$
$$n_j = \frac{k_{2,j}}{k_{1,j}} = \operatorname{ctg}\alpha, \quad 0 < \alpha \leqslant \frac{\pi}{2}. \tag{III.108}$$

Then, according to formula (III.107);

$$\beta_* = -2\arcsin\sqrt{\frac{6\operatorname{ctg}^2\alpha + 1 - \sqrt{8\operatorname{ctg}^2\alpha + 1}}{2(9\operatorname{ctg}^2\alpha + 1)}}. \tag{III.109}$$

Using further equalities (III.106) and (III.108) for determination of the limit values of stresses $q = q_*$, we will obtain

$$q_* = \frac{K\sqrt{2}}{\pi\sqrt{l}} \cdot \frac{1}{\cos^3\frac{\beta_*}{2}\sin^2\alpha\left(1 - 3\operatorname{ctg}\alpha \cdot \operatorname{tg}\frac{\beta_*}{2}\right)}, \tag{III.110}$$

where β_* is determined by expression (III.109).

Thus, as one should have been expected, formulas (III.109) and (III.110) agree with formula (III.53) and (III.54).

2. A plate with a curved crack in the form of a semicircle is extended to infinity by stresses p, acting at an angle α to axis Ox (Fig. 46).

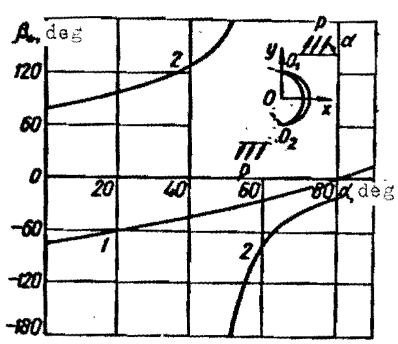


Fig. 46

Considering in formulas (III.100)-(III.102) that $q = 0$, $\Theta = \frac{\pi}{2}$, we find the following expressions for coefficients $k_{1,j}$ and $k_{2,j}$:

$$\left.\begin{aligned}
k^{\bullet}_{1,1} &= \frac{p_* \sqrt{R}}{12\sqrt{2}}[4 - 7\cos 2\alpha + 9\sin 2\alpha];\\
k^{\bullet}_{2,1} &= \frac{p_* \sqrt{R}}{12\sqrt{2}}[4 + 5\cos 2\alpha + 3\sin 2\alpha];\\
k^{\bullet}_{1,2} &= \frac{p_* \sqrt{R}}{12\sqrt{2}}[4 - 7\cos 2\alpha - 9\sin 2\alpha];\\
k^{\bullet}_{2,2} &= -\frac{p_* \sqrt{R}}{12\sqrt{2}}[4 + 5\cos 2\alpha - 3\sin 2\alpha].
\end{aligned}\right\} \tag{III.111}$$

In this case values of parameter n_j ($j = 1, 2$) are determined so:

$$n_1 = \frac{4 + 5\cos 2\alpha + 3\sin 2\alpha}{4 - 7\cos 2\alpha + 9\sin 2\alpha}; \tag{III.112}$$

$$n_2 = -\frac{4 + 5\cos 2\alpha - 3\sin 2\alpha}{4 - 7\cos 2\alpha - 9\sin 2\alpha}, \tag{III.113}$$

where $0 \leq \alpha \leq \frac{\pi}{2}$.

On the basis of formulas (III.107), (III.111), (III.112) and (III.113) it is easy to calculate the value of angle β_*. Curve 1 on Fig. 46 shows the change of the initial direction of crack propagation at end O_1 depending upon angle α, and curve 2 is the dependence of angle β_* on α at end of crack O_2.

Knowing values of angle β_* and using equalities (III.106), (III.111), (III.112) and (III.113), one can determine the limit external stresses $p = p_*$. The dependence of p_* on direction (α) for ends of crack O_1 (curve 1) and O_2 (curve 2) is shown on Fig. 47. From this figure it is clear that the limit stress $p_{*,1}$, necessary for crack to develop at end O_1 is considerably less than $p_{*,2}$, after which the crack starts to spread at end O_2. Thus, when $0 < \alpha < \frac{\pi}{2}$ a crack in the form of an arc of a semicircle starts to spread at end O_1 earlier than at end O_2. The development of a crack at both ends simultaneously is possible in two cases: external stress p acts at an angle $\alpha = 0$ to axis Ox or $\alpha = \frac{\pi}{2}$.

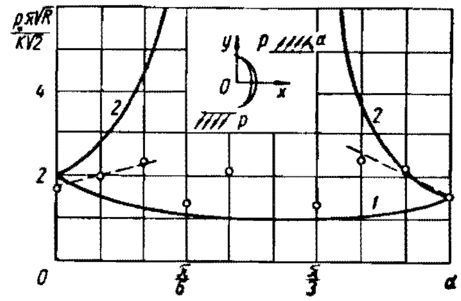


Fig. 47.

3. Experimental investigation of the propagation of a curved crack during extension of a plate. To check the dependence of change of limit stresses $p_{1,*}$ and $p_{2,*}$ for a curved crack in the form of a cut along an arc of the semicircle the following experiments were conducted. From a sheet of silica glass rectangular plates $a \times 2L \times h$ were cut, where $a = 35$ cm, $2L = 20$ cm, and $h = 0.2$ cm. In each plate a glass cutter fastened to a revolving disk shaped a crack-cut in the form of a semicircle with radius $R = 20$ mm, oriented at an angle α to the longitudinal axis of the plate. The plates were extended[7] on a rupture-test machine of MR-0.5 type until in the environments of tip O_1 or O_2 crack propagation began across the cross section of the plate, as a result of which rupture of the plate set in. The value of limit load $Q_{*j}^{(\alpha)}$ was fixed, and then by formula

$$p_{*,j}(\alpha) = \frac{Q_{*,j}^{(\alpha)}}{2Lh} \qquad (j = 1, 2)$$

the value of limit stresses $p_{*,j}(\alpha)$ was calculated.

According to Fig. 47 when $0 < \alpha < \frac{\pi}{2}$ the inequality $p_{*,1}(\alpha) \leq$ $\leq p_{*,2}(\alpha)$ always holds. As theoretical analysis also anticipates during extension of these plates the development of a crack always started from tip O_1. In order for a crack to develop from tip O_2, in tip O_1 a small hole was drilled that blocked crack development from tip O_1, and the crack started to spread from tip O_2 with stresses $p_{*,1}(\alpha) > p_{*,2}(\alpha)$.

Experimental values of forces $Q_{\ ,1}(\alpha)$ [sic] and $Q_{*,2}(\alpha)$ obtained by this method during the extension of glass plates are given in

Table 14. Here $p_*(\alpha)$ are the stresses at an assigned angle α, referred to the mean value of stresses $p_{*,1}(\frac{\pi}{2}) \simeq 0.44$ N/mm^2. The mean values of limit stresses $\frac{p_{*,1}(\alpha)}{p_{*,1}(\frac{\pi}{2})}$ and $\frac{p_{*,2}(\alpha)}{p_{*,1}(\frac{\pi}{2})}$ (circles on Fig. 47) were established by experiment (Table 14) and were multiplied by a coefficient[8] of 1.6.

Table 14.

No. plates	α, rad	$Q^{x}_{*,1} \cdot 10^{-1}$	$Q^{x}_{*,2} \cdot 10^{-1}$	$p_{*\alpha} \cdot 10^{-1}$, N/cm^2	$p^{(av.)}_{*\alpha} / p^{(av.)}_{*\frac{\pi}{2}}$
1	0,000	184	—	0,472	1,05
2	0,000	172	—	0,443	
3	0,000	178	—	0,454	
4	0,174	—	221	0,558	1,27
5	0,174	—	204	0,538	
6	0,174	—	227	0,553	
7	0,349	—	245	0,617	1,43
8	0,349	—	243	0,628	
9	0,349	—	237	0,611	
10	0,524	157	—	0,407	0,85
11	0,524	150	—	0,387	
12	0,524	133	—	0,339	
13	0,524	127	—	0,337	
14	0,698	—	250	0,530	1,38
15	0,698	—	236	0,568	
16	0,698	—	234	0,598	
17	1,047	130	—	0,331	0,85
18	1,047	151	—	0,384	
19	1,047	159	—	0,405	
20	1,047	145	—	0,359	
21	1,222	—	253	0,645	1,50
22	1,222	—	258	0,676	
23	1,222	—	244	0,628	
24	1,396	—	253	0,650	1,41
25	1,396	—	235	0,570	
26	1,396	—	231	0,615	
27	1,571	167	—	0,409	1,00
28	1,571	179	—	0,445	
29	1,571	180	—	0,448	

By comparing theoretical and experimental data, it is possible to note that theoretical results will qualitatively agree with the results of experiment.

4. Influence on limit load of distortion on the contour of a crack. Let us compare the values of limit stresses for a plate weakened by a curved crack in the form of a cut along the arc of the circumference (see Fig. 48, where R is the radius, and 2Θ the central angle of the curved crack), when the plate is subjected to uniaxial extension by a field of uniform stresses p, directed along the line of symmetry of the cut-crack, and for the same plate, but weakened by a linear macrocrack of length $2l = 2R \sin \Theta$ (dotted line on Fig. 48). For the second problem, i.e., for a plate with a linear macrocrack of length $2R \sin \Theta$, limit stresses are expressed by the Griffith formula:

$$p_*^{(r)} = \frac{K\sqrt{2}}{\pi\sqrt{R\sin\Theta}}. \tag{III.114}$$

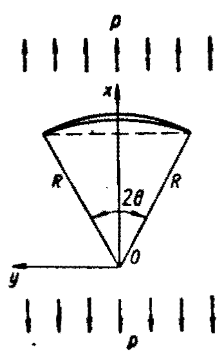


Fig. 48.

For the first problem the value of limit stresses $p = p_*$ is found by formula (III.104), (III.106) and (III.107). Considering in (III.104) that $\alpha = 0$, we find

$$k_{1,1} = \frac{p\sqrt{R\sin\Theta}}{2}\left(\frac{1-\sin^2\frac{\Theta}{2}\cos^2\frac{\Theta}{2}}{1+\sin^2\frac{\Theta}{2}}\cos\frac{\Theta}{2} + \cos\frac{3}{2}\Theta\right); \tag{III.115}$$

$$k_{2,1} = \frac{p\sqrt{R\sin\Theta}}{2}\left(\frac{1-\sin^2\frac{\Theta}{2}\cdot\cos^2\frac{\Theta}{2}}{1+\sin^2\frac{\Theta}{2}}\sin\frac{\Theta}{2} + \sin\frac{3}{2}\Theta\right).$$

Moreover, according to equality (III.102) we have

$$k_{1,2} = k_{1,1}; \quad k_{2,2} = -k_{2,1}.$$

Hence on the basis of equations (III.106) and (III.107) we obtain

$$p_* = \frac{K\sqrt{2}}{\pi\sqrt{R\sin\Theta}} \cdot \frac{8}{f(\Theta)}. \tag{III.116}$$

Here

$$f(\Theta) = \tilde{k}_{1,1}\left(3\cos\frac{\beta_*}{2} + \cos\frac{3}{2}\beta_*\right) - 3\tilde{k}_{2,1}\left(\sin\frac{\beta_*}{2} + \sin\frac{3}{2}\beta_*\right), \tag{III.117}$$

where

$$\tilde{k}_{1,1} = \frac{2k_{1,1}}{p\sqrt{R\sin\Theta}}; \quad \tilde{k}_{2,1} = \frac{2k_{2,1}}{p\sqrt{R\sin\Theta}};$$

$$\beta_* = -2\arcsin\sqrt{\frac{6n_1^2 + 1 - \sqrt{8n_1^2 + 1}}{2(9n_1^2 + 1)}}; \quad n_1 = \frac{k_{2,1}}{k_{1,1}}$$

(values of $k_{1,1}$ and $k_{2,1}$ are determined by formulas (III.115).

Comparing equality (III.114) and (III.116), we easily find

$$\frac{p_*}{p_*^{(r)}} = \frac{8}{f(\Theta)}, \tag{III.118}$$

where function f(Θ) is determined by equality (III.117).

On Fig. 49 (solid line) the change of relationship (III.118) when $0 < \Theta < \frac{\pi}{2}$ is graphed. The deflection of this line from one characterizes the influence of distortion of the contour of crack on its limit load. If the central angle of curved crack 2Θ is such that $\Theta \leq \frac{\pi}{3}$, then, Fig. 49 shows the limit values of external stresses calculated by formulas (III.114) or (III.116) are practically the same.

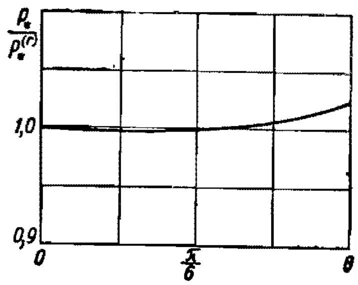


Fig. 49.

Certain additional data about the influence of distortion of the contour of an initial crack its propagation have been published in work [134, 136, 215, 217].

Studies of elastic equilibrium of a nonuniform plate with cuts along the arcs of the circumference can be found in works [41, 42, 145, 146, 151, 166].

Footnotes

[1]A detailed account of the two-dimensional problem of the theory elasticity is given in the monograph of N. I. Muskhelishvili [103] (see also the book of G. N. Savin [150], L. A. Galin [33], M. P. Sheremet'yev [170], S. I. Belonosov [17]).

[2]This hypothesis is advanced in works [22, 131]. Analogous assumptions are advanced also in works [167, 185].

[3]In this case in the plane of location of the crack components of the stress tensor are expressed so:

$$\sigma_y(x,\,0) = p\sin^2\alpha + q\cos^2\alpha,\quad \sigma_x(x,\,0) = p\cos^2\alpha + q\sin^2\alpha,\quad \tau_{yx}(x,\,0) = (p-q)\sin\alpha\cos\alpha.$$

[4]Above an analysis is given of the maximum-equilibrium state of a crack with zero radius of curvature in its tip, i.e., the case of a peaked crack – a cavity – is examined.

[5]The plane state of strain in tubular samples is achieved by the imposition axial extension-compression q and internal pressure p in different relationships ($q/p = \eta_0$).

[6]Some of these investigations are expounded in subsequent sections.

[7]The cracks were formed and the specimens fixed in the clamps of the machine just as described in Section 4 of this chapter.

[8]
$$p_*(\alpha) = \frac{\sqrt{2}K}{\pi\sqrt{R}}f(\alpha);\quad f\left(\frac{\pi}{2}\right) = 1{,}6.$$

CHAPTER IV

MAXIMUM EQUILIBRIUM OF PLATES WEAKENED BY PEAKED HOLES

1. Introduction

Below certain problems are examined about the maximum-equilibrium state of plates weakened by holes with angular points (cusps) on the contour. The solution of such problems is of interest both from the point of view of clarifying the influence of similar defects on the supporting power of deformable brittle bodies and also for determining distinctions in values of limit loads for a body weakened by a hole with sharp (crack-like) ends, and a body weakened by a linear crack of corresponding length.

The course of solution of such problems is similar to that given above: in the beginning under an assigned external influence we determine the field of elastic stresses in the environment of the sharp ends of the examined hole, and then, using equations (III.34) and (III.35), we find the limit load for the problem at hand.

However, it is necessary to note that for determination of the field of elastic stresses in a plate bounded by a contour with angular points, direct use of the method of N. I. Muskhelishvili [103] in general is difficult.[1]

In certain cases the problem of the theory of elasticity for a region bounded by a contour with angular pears can be solved also by passage to the limit in the solution for the region bounded by a

smooth contour (depending on certain parameters), when this contour tightens into a peaked contour. For example, the problem for a plate with a linear cut (crack) can be solved [103] by passage to the limit in the solution to the problem of theory of elasticity for a plate with an elliptical hole, and the problem of theory of elasticity for a plate with similinfinite cut can be solved by passage to the limit in the solution of the corresponding problem for the region outside the parabolic contour. Certain other such examples will be given below.

2. Extension of a Plate with a Hypocycloid-Shaped Hole

Let us examine an infinite plate of unit thickness, weakened by a hypocycloid-shaped hole, i.e., a cut whose boundary (contour) L is described in plane xOy by the equations

$$x = A\left(\cos\vartheta + \frac{1}{n}\cos n\vartheta\right); \quad y = A\left(\sin\vartheta - \frac{1}{n}\sin n\vartheta\right), \qquad \text{(IV.I)}$$

where $0 \leqslant \vartheta \leqslant 2\pi$; $A = \frac{na}{n+1}$; $a > 0$; n is a positive integer.

On Figs. 50 and 51 according to equations (IV.I) contours L are depicted respectively when n = 2 (hypocycloid with three tips) and n = 3 (hypocycloid with four tips). The tips of a hypocycloid as directly follows from the equation, are angular cusps and, consequently, for a fixed value of n contour L has n + 1 cusps.

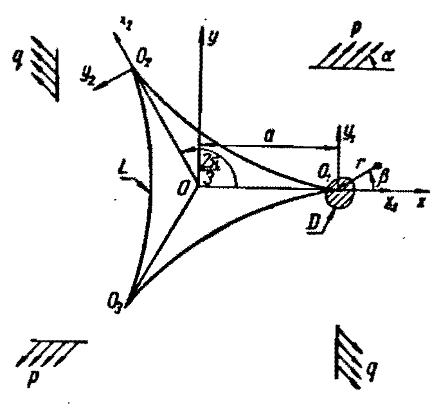


Fig. 50.

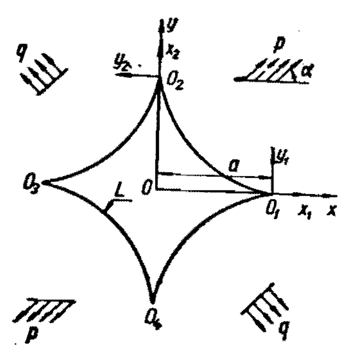


Fig. 51.

Let us assume that a plate with a hole in the form of a hypocycloid is subjected to extension at infinity by external stresses p and q, acting in mutually perpendicular directions. Moreover, stresses p are directed at an angle α to axis Ox. The contour of the hole is considered free from external stresses. Let us define the stressed-deformed state in the plate, i.e., let us find for the problem functions $\varphi_1(z)$ and $\psi_1(z)$.

Contour L is free from external stresses, therefore for the determination of functions $\varphi_1(z)$ and $\psi_1(z)$ we have the following contour condition [103]:

$$\varphi_1(z) + z\overline{\varphi_1'(z)} + \overline{\psi_1(z)} = 0 \text{ on } L \qquad \text{(IV.2)}$$

<u>Auxiliary problem</u> (hole in the form of a hypotrochoid). Let us consider the function

$$z = \omega(\zeta) = A_1\left(\zeta + \frac{m}{\zeta^n}\right); \quad A_1 = \frac{a}{1+m}, \qquad \text{(IV.3)}$$

where $z = x + iy$; $\zeta = \xi + i\eta = \rho e^{i\theta}$; $0 \leqslant m \leqslant \frac{1}{n}$; n is a positive integer.

This function realizes conformal mapping of the exterior of unit circle $C_1(\rho = 1)$ in plane ζ onto the exterior of a hypotrochoid in plane z. It is easy to verify that when $m = \frac{1}{n}$ function (IV.3) depicts the exterior of a unit circle in plane ζ onto the exterior of a hypocycloid. In this case we obtain contour L in plane z from equation (IV.3) when $m = \frac{1}{n}$ and $|\zeta| = 1$.

Thus, if under assigned conditions in points at infinity of a plate with a hole in the form of a hypotrochoid a problem of the theory of elasticity is solved, the solution of the corresponding problem of the theory of elasticity for a plate with a hole in the form of a hypocycloid with the $(n+1)$-th cusp on the contour is obtained by considering $m = \frac{1}{n}$. For now we take $mn < 1$.

Contour condition (IV.2) during conformal mapping (IV.3) takes the form

$$\varphi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = 0, \text{ on } C_1 \tag{IV.4}$$

where

$$\varphi(\zeta) = \varphi_1[\omega(\zeta)]; \quad \psi(\zeta) = \psi_1[\omega(\zeta)]; \quad \sigma = e^{i\vartheta};$$
$$\varphi_1'(z) = \frac{\varphi'(\zeta)}{\omega'(\zeta)}; \quad \psi_1'(z) = \frac{\psi'(\zeta)}{\omega'(\zeta)};$$

function $\omega(\zeta)$ is determined by equality (IV.3).

Using expression (IV.3), we find

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \frac{1}{\sigma^n} \cdot \frac{\sigma^{n+1}+m}{1-mn\sigma^{n+1}}; \quad \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} = \sigma^n \frac{1+m\sigma^{n+1}}{\sigma^{n+1}-mn}. \tag{IV.4a}$$

For the examined infinite region with a hole functions $\varphi(\zeta)$ and $\psi(\zeta)$ have the following form:

$$\varphi(\zeta) = A_1\Gamma\zeta + \varphi_0(\zeta); \quad \psi(\zeta) = A_1\Gamma'\zeta + \psi_0(\zeta), \tag{IV.5}$$

where $\psi_0(\zeta)$ and $\varphi_0(\zeta)$ are functions holomorphic in region $|\zeta| \geqslant 1$, outside unit circle C_1. For definitiveness it is possible to consider that $\varphi_0(\infty) = 0$, and Γ and Γ' are constants determining the state of strain in points at infinity of plates (see Chap. III),

$$\Gamma = \overline{\Gamma} = \frac{1}{4}(p+q); \quad \Gamma' = -\frac{1}{2}(p-q)e^{-2i\alpha}. \tag{IV.5a}$$

Thus, on the basis of equalities (IV.4) and (IV.5) we have

$$\varphi_0(\sigma) + \frac{\sigma^{n+1}+m}{\sigma^n(1-mn\sigma^{n+1})} \overline{\varphi_0'(\sigma)} + \overline{\psi_0(\sigma)} = f_0(\sigma), \tag{IV.6}$$

where $mn < 1$; $A_1 > 0$;

$$f_0(\sigma) = -A_1\Gamma\left[\sigma + \frac{\sigma^{n+1}+m}{\sigma^n(1-mn\sigma^{n+1})}\right] - \frac{A_1\overline{\Gamma'}}{\sigma}. \tag{IV.6a}$$

Function $\psi_0(\sigma)$ is the boundary value of function $\psi_0(\zeta)$, holomorphic outside unit circle C_1. Then on the basis of certain therorems [103], from contour condition (IV.6) we find

$$\varphi_0(\zeta) = \frac{-1}{2\pi i}\int_{C_1}\frac{f_0(\sigma)\,d\sigma}{\sigma-\zeta} + \frac{1}{2\pi i}\int_{C_1}\frac{\sigma^{n+1}+m}{\sigma^n(1-mn\sigma^{n+1})}\overline{\varphi_1'(\sigma)}\frac{d\sigma}{\sigma-\zeta}, \qquad \text{(IV.7)}$$

where $|\zeta| \geqslant 1$.

Further, inasmuch as $\varphi_0(\sigma)$ is the boundary value of function $\varphi_0(\zeta)$, holomorphic outside circumference C_1, then from contour condition (IV.6) we will obtain

$$\psi_0(\zeta) = -\frac{1}{2\pi i}\int_{C_1}\frac{\overline{f_0(\sigma)}}{\sigma-\zeta}\,d\sigma + \frac{1}{2\pi i}\int_{C_1}\frac{\sigma^n(1+m\sigma^{n+1})}{\sigma^{n+1}-mn}\times$$

$$\times\,\varphi_\sigma'(\sigma)\frac{d\sigma}{\sigma-\zeta} + \psi_0(\infty) \qquad \text{(IV.8)}$$

at $|\zeta| \geqslant 1$.

Consequently, if integrals in the right side of formulas (IV.7) and (IV.8), will be definite, then also functions $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ will be definite. Constant $\psi_0(\infty)$ in formula (IV.8) can be omitted, since it does not affect the distribution of stresses in the plate.

For calculation of integrals

$$I_1(\zeta) = \frac{1}{2\pi i}\int_{C_1}\frac{\sigma^{n+1}+m}{\sigma^n(1-mn\sigma^{n+1})}\overline{\varphi_0'(\sigma)}\frac{d\sigma}{\sigma-\zeta}; \qquad \text{(IV.9)}$$

$$I_2(\zeta) = \frac{1}{2\pi i}\int_{C_1}\frac{\sigma^n(1+m\sigma^{n+1})}{\sigma^{m+1}-mn}\varphi_0'(\sigma)\frac{d\sigma}{\sigma-\zeta} \qquad \text{(IV.10)}$$

let us examine the following two functions of complex variable ζ:

$$\frac{1}{\zeta^n}\cdot\frac{\zeta^{n+1}+m}{1-mn\zeta^{n+1}}\overline{\varphi_0'}\left(\frac{1}{\zeta}\right) \text{ and } \frac{\zeta^n(1+m\zeta^{n+1})}{\zeta^{n+1}-mn}\varphi_0'(\zeta), \qquad \text{(IV.11)}$$

where when $|\zeta| > 1$

$$\varphi_0(\zeta) = \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \ldots,$$
$$\varphi_0'(\zeta) = -\frac{a_1}{\zeta^2} - \frac{2a_2}{\zeta^3} - \ldots;$$

when $|\zeta| < 1$

$$\overline{\varphi_0'}\left(\frac{1}{\zeta}\right) = -\bar{a}_1\zeta^2 - 2\bar{a}_2\zeta^3 - \ldots.$$

It is easy to note that the integrand in $I_1(\zeta)$ is the boundary value on the contour of unit circle C_1 of the first of functions (IV.11), and the integrand in $I_2(\zeta)$ is the boundary value on circumference C_1, of the second of functions (IV.11).

Function

$$\frac{1}{\zeta^n} \cdot \frac{\zeta^{n+1} + m}{1 - mn\zeta^{n+1}} \overline{\varphi_0'}\left(\frac{1}{\zeta}\right)$$

is holomorphic inside circumference C_1, with the exception of point $\zeta = 0$, where it has a pole with the principal part

$$G_0(\zeta) = \begin{cases} -m\left(\frac{\bar{a}_1}{\zeta^{n-2}} + 2\frac{\bar{a}_2}{\zeta^{n-3}} + \ldots + \frac{(n-2)\bar{a}_{n-2}}{\zeta}\right) \\ \quad \text{when } n > 2; \\ 0 \quad \text{when } n \leqslant 2, \end{cases} \tag{IV.12}$$

where n is a positive integer.

Consequently, according to data of [103], the integral

$$I_1(\zeta) = -G_0(\zeta) \text{ when } |\zeta| \geqslant 1, \tag{IV.13}$$

where $G_0(\zeta)$ is determined by equality (IV.12).

The second of functions (IV.11) is holomorphic outside the circumference with the exception of point $\zeta = \infty$, where it has a pole with the principle part

$$G_\infty(\zeta) = \begin{cases} -m\,[a_1\zeta^{n-2} + 2a_2\zeta^{n-3} + \ldots + (n-1)\,a_{n-1}] \\ \quad \text{when } n \geqslant 2; \\ 0 \quad \text{when } n < 2. \end{cases} \tag{IV.14}$$

Therefore

$$I_2(\zeta) = -\frac{\zeta^n(1 + m\zeta^{n+1})}{\zeta^{n+1} - mn}\,\varphi_0'(\zeta) + G_\infty(\zeta) \text{ when } |\zeta| \geqslant 1. \tag{IV.15}$$

Thus, on the basis of formulas (IV.7)-(IV.10), (IV.13) and (IV.15) we have

$$\begin{gathered} \varphi_0(\zeta) = -\frac{1}{2\pi i}\int\limits_{C_1}\frac{f_0(\sigma)}{\sigma-\zeta}\,d\sigma - G_0(\zeta); \\ \psi_0(\zeta) = -\frac{1}{2\pi i}\int\limits_{C_1}\frac{\overline{f_0(\sigma)}\,d\sigma}{\sigma-\zeta} - \frac{\zeta^n(1 + m\zeta^{n+1})}{\zeta^{n+1} - mn}\,\varphi_0'(\zeta) + G_\infty(\zeta), \end{gathered} \tag{IV.16}$$

where $|\zeta| \geqslant 1$, $f_0(\sigma)$, $G_0(\zeta)$ and $G_\infty(\zeta)$ are determined accordingly by formulas (IV.6a), (IV.12) and (IV.14).

Calculating the integrals appearing in formulas (IV.16), we obtain

$$\begin{gathered} \frac{1}{2\pi i}\int\limits_{C_1}\frac{f_0(\sigma)\,d\sigma}{\sigma-\zeta} = \frac{A_1\Gamma m}{\zeta^n} + \frac{A_1\overline{\Gamma'}}{\zeta} \qquad (|\zeta| \geqslant 1); \\ \frac{1}{2\pi i}\int\limits_{C_1}\frac{\overline{f_0(\sigma)}\,d\sigma}{\sigma-\zeta} = \frac{A_1\Gamma}{\zeta} + \frac{(1 + nm^2)\,\zeta^n A_1\Gamma}{\zeta^{n+1} - mn} \qquad (|\zeta| \geqslant 1). \end{gathered}$$

Hence, according to formulas (IV.5) and (IV.16) we have

$$\left.\begin{gathered} \varphi(\zeta) = A_1\Gamma\zeta + \varphi_0(\zeta); \\ \varphi_0(\zeta) = -G_0(\zeta) - A_1\left(\frac{m\Gamma}{\zeta^n} + \frac{\overline{\Gamma'}}{\zeta}\right); \\ \psi(\zeta) = A_1\Gamma'\zeta + G_\infty(\zeta) - \frac{\zeta^n(1 + m\zeta^{n+1})}{\zeta^{n+1} - mn}\,\varphi_0'(\zeta) - A_1\Gamma\left[\frac{1}{\zeta} + \right. \\ \left. + \frac{(1 + m^2n)\,\zeta^n}{\zeta^{n+1} - mn}\right], \end{gathered}\right\} \tag{IV.17}$$

where $|\zeta| \geqslant 1$, $G_0(\zeta)$ and $G_\infty(\zeta)$ are determined accordingly by formulas (IV.12) and (IV.14); $n = 1, 2, 3, \ldots$; $m \leqslant 1$.

From equalities (IV.12) and (IV.14); $n = 1, 2$ functions $\varphi(\zeta)$ and $\psi(\zeta)$ can be determined by formulas (IV.17). At $n > 2$ in formulas (IV.17) will appear unknown coefficients a_n $(n = 1, 2, 3 \ldots)$. At $|\zeta| \geqslant 1$ we have

$$\varphi_0(\zeta) = \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \frac{a_3}{\zeta^3} + \ldots + \frac{a_n}{\zeta^n} + \ldots$$

Equating this expression to the expression for $\varphi_0(\zeta)$ in formulas (IV.17), at $n > 2$

$$a_1 = -\frac{A_1\overline{\Gamma'}}{1 - m^2(n-2)}; \quad a_{n-2} = -\frac{A_1 m\overline{\Gamma'}}{1 - m^2(n-2)}; \quad a_n = -mA_1\Gamma.$$

Then on the basis of formulas (IV.17) at $n > 2$ we have

$$\begin{gathered}\varphi(\zeta) = A_1\left\{\Gamma\left(\zeta - \frac{m}{\zeta^n}\right) - \frac{1}{[1 - m^2(n-2)]\,\zeta^{n-2}}\left(m\Gamma' + \zeta^{n-3}\overline{\Gamma'}\right)\right\}, \\ A_1 = \frac{a}{1+m}; \\ \psi(\zeta) = A_1\Gamma'\zeta - \frac{A_1(1 + nm^2)}{\zeta^{n+1} - nm}\left\{2\Gamma\zeta^n + \frac{1}{1 - m^2(n-2)} \times \right. \\ \left. \times\, [(n-2)m\Gamma'\zeta + \Gamma'\zeta^{n-2}]\right\}.\end{gathered} \quad \text{(IV.18)}$$

Considering in formulas (IV.17) that $n = 1$, $m < 1$, we obtain certain [103] complex potentials $\varphi(\zeta)$ and $\psi(\zeta)$ for plates with an elliptic hole, in particular, at $m = 0$ for plates with a circular hole.

If in formulas (IV.17) we set $m = \frac{1}{n} = 1$, it is easy to find functions $\varphi(\zeta)$ and $\psi(\zeta)$ for a plate with a linear cut (crack), i.e., for a plate with a hole whose contour has two angular cusps.

Determination of functions $\varphi(\zeta)$ and $\psi(\zeta)$ for a plate with a hole in the form of a hypocycloid. Considering in formulas (IV.12), (IV.14) and (IV.17) that $m = \frac{1}{n} \leqslant 1$ $(m \geqslant 0)$, we obtain (as in the case of a plate with linear cut) functions $\varphi(\zeta)$ and $\psi(\zeta)$, characterizing the state of strain in a plate with a hole in the form of a hypocycloid with the $(n+1)$-th cusp on its contour.

In this case

$$\varphi(\zeta) = A\Gamma\zeta - G_0^\bullet(\zeta) - A\left(\frac{\Gamma}{n\zeta^n} + \frac{\overline{\Gamma'}}{\zeta}\right);$$

$$\psi(\zeta) = A\Gamma'\zeta + G_\infty^\bullet(\zeta) - \frac{\zeta^n\left(1 + \frac{1}{n}\zeta^{n+1}\right)}{\zeta^{n+1} - 1}\varphi_0'(\zeta) - \tag{IV.19}$$

$$- A\Gamma\left[\frac{1}{\zeta} + \frac{\left(1 + \frac{1}{n}\right)\zeta^n}{\zeta^{n+1} - 1}\right],$$

where

$$\varphi_0(\zeta) = -G_0^\bullet(\zeta) - A\left(\frac{\Gamma}{n\zeta^n} + \frac{\overline{\Gamma'}}{\zeta}\right), \quad A = \frac{na}{n+1}, \quad a > 0;$$

$$G_0^\bullet(\zeta) = \begin{cases} -\frac{1}{n}\left(\frac{\bar{a}_1}{\zeta^{n-2}} + 2\frac{\bar{a}_2}{\zeta^{n-3}} + \ldots + \frac{(n-2)\,\bar{a}_{n-2}}{\zeta}\right) & \text{when } n > 2; \\ 0 \text{ when } n \leqslant 2; \end{cases}$$

$$G_\infty^\bullet(\zeta) = \begin{cases} -\frac{1}{n}\left[a_1\zeta^{n-2} + 2a_2\zeta^{n-3} + \ldots + (n-1)\,a_{n-1}\right] & \text{when } n \geqslant 2; \\ 0 \text{ when } n < 2 \qquad (n = 1, 2, 3, \ldots). \end{cases}$$

At $n > 2$ and $m = \frac{1}{n}$ formulas (IV.18) take the following form:

$$\varphi(\zeta) = \frac{na}{n+1}\left\{\Gamma\left(\zeta - \frac{1}{n\zeta^n}\right) - \right.$$

$$\left. - \frac{n}{(n^2 - n + 2)\,\zeta^{n-2}}\left(\Gamma' + n\overline{\Gamma'}\zeta^{n-3}\right)\right\},$$

$$\psi(\zeta) = \frac{na}{n+1}\Gamma'\zeta - \frac{a}{\zeta^{n+1} - 1}\left\{2\Gamma\zeta^n + \right. \tag{IV.20}$$

$$\left. + \frac{n}{n^2 - n + 2}\left[(n-2)\,\Gamma'\zeta + n\overline{\Gamma'}\zeta^{n-2}\right]\right\}.$$

Below cases of extension of an infinite plate with a hole in the form of a hypocycloid with three tips $(n = 2)$ and in the form of an astroid, i.e., hypocycloid with four tips $(n = 3)$ are examined in detail. Let us note that the elastic equilibrium of a plate with a hole in the form of an astroid is examined by other means in a work of S. M. Belonosov [17].

For these special cases according to formulas (IV.19) and (IV.20) functions $\varphi(\zeta)$ and $\psi(\zeta)$ have the following form:

I. At $n = 2$ (see Fig. 50):

$$\varphi(\zeta) = \frac{2}{3} a \left\{ \Gamma\left(\zeta - \frac{1}{2\zeta^2}\right) - \frac{\overline{\Gamma'}}{\zeta} \right\} \quad (|\zeta| \geqslant 1);$$
$$\psi(\zeta) = \frac{2}{3} a \left\{ \Gamma'\zeta + \frac{3\zeta^2}{\zeta^3 - 1} \Gamma - \frac{2 + \zeta^3}{2(\zeta^3 - 1)} \overline{\Gamma'} \right\} - \frac{a_1}{2}, \tag{IV.21}$$

where $a > 0$. In the formula for $\psi(\zeta)$ additive component $-\frac{a_1}{2}$ can be removed, since it does not affect the distribution of stresses in the plate.

II. At $n = 3$ (see Fig. 51):

$$\varphi(\zeta) = \frac{3a}{4} \left\{ \Gamma\zeta - \frac{\Gamma}{3\zeta^3} - \frac{\overline{\Gamma'}}{\zeta} \right\} + \frac{\overline{a_1}}{3\zeta} \quad (|\zeta| > 1);$$
$$\psi(\zeta) = \frac{3a}{4} \Gamma'\zeta - \frac{a}{\zeta^4 - 1} \left[2\Gamma\zeta^3 - \frac{4}{3} \frac{a_1}{a} \zeta \right], \tag{IV.22}$$

where

$$a_1 = -\frac{9a}{32} (\Gamma' + 3\overline{\Gamma'}). \tag{IV.23}$$

3. Determination of Elastic Stresses in the Neighborhood of Angular Points of a Plate with a Hypocycloid-Shaped Hole

Case of a hole in the form of a hypocycloid with three tips. To determine stresses in the neighborhood of angular points (cusps) of the boundary of the examined plate (see Fig. 50) we use certain Kalosov-Muskhelishvili formulas [103], expressing components of stress in corresponding curvilinear coordinates ρ, ϑ through complex potentials $\Phi(\zeta)$ and $\Psi(\zeta)$, i.e., we use the formulas

$$\sigma_\rho + \sigma_\vartheta = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}] \quad (\zeta = \rho e^{i\vartheta}); \tag{IV.24}$$

$$\sigma_\vartheta - \sigma_\rho + 2i\tau_{\rho\vartheta} = \frac{2\zeta^2}{\rho^2 \overline{\omega'(\zeta)}} \{\overline{\omega(\zeta)} \Phi'(\zeta) + \omega'(\zeta) \Psi(\zeta)\}. \tag{IV.25}$$

Here $\sigma_\rho, \sigma_\vartheta, \tau_{\rho\vartheta}$ are components of the elastic stress tensor in curvilinear system of coordinates (ρ), (ϑ); if in a given point of the body we take moving rectilinear, rectangular coordinates $x'O'y'$ such that axis $O'x'$ coincides with axis (ρ), where (ρ) is tangent to line $\vartheta = \text{const}$, and axis $O'y'$ with axis ϑ, then

$$\sigma_\rho = \sigma_{x'}; \quad \sigma_\vartheta = \sigma_{y'}; \quad \tau_{\rho\vartheta} = \tau_{x'y'}; \tag{IV.26}$$

$\Phi(\zeta)$ and $\Psi(\zeta)$ — complex potentials connected with functions $\varphi(\zeta)$ and $\psi(\zeta)$ by relationships

$$\Phi(\zeta) = \frac{\varphi'(\zeta)}{\omega'(\zeta)}; \quad \Psi(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)}, \tag{IV.27}$$

where $\omega(\zeta)$ — function of conformal mapping, $z = \omega(\zeta)$.

For our problem functions $\varphi(\zeta)$ and $\psi(\zeta)$ are represented by formulas (IV.21), and the connection between variables z and ζ is carried out by formula (IV.3) if in it we set $m = \frac{1}{n}$ and $n = 2$, i.e.

$$z = \omega(\zeta) = \frac{2}{3} a \left(\zeta + \frac{1}{2\zeta^2}\right). \tag{IV.28}$$

According to formulas (IV.21) and(IV.26)-(IV.28)

$$\Phi(\zeta) = \frac{1}{\zeta^3 - 1} \{(\zeta^3 + 1)\Gamma + \zeta\bar{\Gamma}'\}, \quad \omega'(\zeta) = \frac{2}{3} a \frac{\zeta^3 - 1}{\zeta^3}; \tag{IV.29}$$

$$\Phi'(\zeta) = \frac{-1}{(\zeta^3 - 1)^2} \{6\zeta^2\Gamma + 2\zeta^3\bar{\Gamma}' + \bar{\Gamma}'\};$$

$$\Psi(\zeta) = \frac{1}{\zeta^3 - 1} \left\{\zeta^3\Gamma' + \frac{3\zeta^4(2 + \zeta^3)}{(\zeta^3 - 1)^2}\Gamma + \frac{9}{2} \frac{\zeta^5}{(\zeta^3 - 1)^2}\bar{\Gamma}'\right\}.$$

Subsequently we will be interested in distribution of elastic stresses in a plate in a small neighborhood of angular points (z_{0j}, j = 1, 2, 3) of the contour of hole, therefore for convenience of calculations it is useful to introduce either new system of rectangular cartesian coordinates $x_j O_j y_j$ (j = 1, 2, 3) with origin in angular point (O_j) and oriented so that axis $O_j x_j$ coincides with the bisector of the angle in the tip of an angular point and is directed to the inside

of the plate, or to introduce polar system of coordinates (r, β) with origin at point z_{0j} and polar axis $O_j x_j$. Let us note also that for our example angular points O_j $(j = 1, 2, 3)$ are determined in plane $z = x + iy$ (see Fig. 50) by the following coordinates z_{0j} $(j = 1, 2, 3)$:

$$z_{01} = a + i0; \quad z_{02} = -\frac{a}{2}(1 - i\sqrt{3});$$
$$z_{03} = -\frac{a}{2}(1 + i\sqrt{3}). \tag{IV.30}$$

For these points axis $O_j x_j$ $(j = 1, 2, 3)$ will form with axis Ox angle ϑ_j $(j = 1, 2, 3)$, where $\vartheta_1 = 0$, $\vartheta_2 = \frac{2}{3}\pi$, $\vartheta_3 = -\frac{2}{3}\pi$ for the first 0_1, second 0_2 and third 0_3 tips of the examined hypocycloid respectively.

According to conformal mapping (IV.28), to these points in plane $\zeta = \xi + i\eta$ (Fig. 52) correspond the following points of a unit circle:

$$\sigma_1 = e^{i0}, \quad \sigma_2 = e^{i\frac{2}{3}\pi}, \quad \sigma_3 = e^{-i\frac{2}{3}\pi},$$

or

$$\sigma_j = e^{i\vartheta_j} \qquad (j = 1, 2, 3). \tag{IV.31}$$

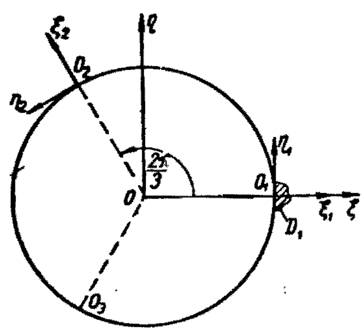


Fig. 52.

Let us examine now small region D of the plate (Fig. 50) in the neighborhood of angular point z_{0j} $(j = 1, 2, 3)$. Affixes z and z_j of points of this region accordingly in system of coordinates xOy and $x_jO_jy_j$ are interconnected by the equality

$$z = z_{0j} + z_j e^{i\vartheta_j} \quad (|z_j| \ll a). \tag{IV.32}$$

According to conformal mapping (IV.28), small region D of plane (see Fig. 50) in the neighborhood of angular point z_{0j} passes into certain small region D_1 in plane ζ (see Fig. 52) in the neighborhood corresponding point σ_j $(j = 1, 2, 3)$. Moreover, the connection between affixes ζ and ζ_j by points of region D_1 can be written as:

$$\zeta = \sigma_j + \zeta_j e^{i\theta_j} \quad (\zeta_j = \rho_j e^{i\lambda}, \quad \rho_j \ll 1), \tag{IV.33}$$

where ζ is a complex variable in system of coordinates $\xi O \eta$, and ζ_j in system of coordinates $\xi_j O_j \eta_j$ with origin at point σ_j; values of σ_j are determined by equality (IV.31).

Putting equalities (IV.32) and (IV.33) into (IV.28) and expanding the right side of the obtained expression in a series in powers ζ_j $(|\zeta_j| \ll 1)$, we find

$$\sigma_j^3 z_j = a\zeta_j^2 + 0(|\zeta_j|^3) \qquad (j = 1, 2, 3).$$

Hence and on the basis of equality (IV.31) accurate to small $(|\zeta_j|^3)$ we have the formula

$$\zeta_j = \sqrt{\frac{z_j}{a}} \qquad (z_j = re^{i\beta}, \qquad j = 1, 2, 3), \tag{IV.34}$$

where the radical sign is determined by the condition $\mathrm{Re}\,\zeta_i > 0$.

In the case of the n-angular hole an analogous formula can be written so:

$$\zeta_j = \sqrt{\frac{2z_j}{na}} \qquad (n = 2, 3, 4 \ldots). \tag{IV.34a}$$

Formula (IV.32)-(IV.34) will be used later in calculating the field of elastic stresses in the neighborhood of angular points of the examined region, and also at the transition into functions $\varphi(\zeta)$, $\Phi(\zeta)$ and $\psi(\zeta)$, $\Psi(\zeta)$ from variables ζ to variables z in the neighborhood of angular points z_{0j}.

On the basis of formulas (IV.24), (IV.25), (IV.28) and (IV.29) for coordinate plane $\vartheta = 0$ $(\zeta = \rho)$

$$\sigma_\rho + \sigma_\vartheta = \frac{2}{\rho^3 - 1}\{2(\rho^3 + 1)\Gamma + \rho(\overline{\Gamma}' + \Gamma')\}; \tag{IV.35}$$

$$\sigma_\vartheta - \sigma_\rho + 2i\tau_{\rho\vartheta} = \frac{2\rho^3}{\rho^3 - 1}\Big\{\Gamma' + \frac{1}{(\rho^3 - 1)^2}[(6\rho + 3\rho^4 - 6\rho^3 - 3)\Gamma + \\ + \frac{1}{2\rho^2}(9\rho^4 - 4\rho^6 - 4\rho^3 - 1)\overline{\Gamma}']\Big\}. \tag{IV.36}$$

For small neighborhood D_1 (see Fig. 50) at $\vartheta = 0$ values of ρ according to equality (IV.29) can be represented so: $\rho = 1 + \rho_1$, where $\rho_1 \ll 1$. Putting in formulas (IV.35) and (IV.36) values of $\rho = 1 + \rho_1$ and expanding the obtained expressions in a series in powers of ρ_1, we find

$$\begin{aligned} \sigma_\rho + \sigma_\vartheta &= \frac{2}{\rho^3 - 1}\{2(2 + 3\rho_1 + 3\rho_1^2 + \rho_1^3)\Gamma + \\ &\quad + (1 + \rho_1)(\overline{\Gamma}' + \Gamma')\}, \\ \sigma_\vartheta - \sigma_\rho + 2i\tau_{\rho\vartheta} &= \frac{2\rho^3}{\rho^3 - 1}\Big\{\Gamma' + \frac{1}{(\rho^3 - 1)^2}[(6\rho_1^3 + 3\rho_1^4)\Gamma - \\ &\quad - (9\rho_1^2 + 6\rho_1^3 + 0(\rho_1^4))\overline{\Gamma}']\Big\}. \end{aligned} \tag{IV.37}$$

We note further that

$$\frac{1}{\rho^3 - 1} = \frac{1}{3\rho_1}\left(1 - \rho_1 + \frac{2}{3}\rho_1^2 - \frac{1}{3}\rho_1^3 + 0(\rho_1^4)\right),$$

$$\frac{1}{(\rho^3 - 1)^2} = \frac{1}{9\rho_1^2}\left(1 - 2\rho_1 + \frac{7}{3}\rho_1^2 - 2\rho_1^3 + 0(\rho_1^4)\right).$$

On the basis of these equalities formulas (IV.37) are easily transformed to the form

$$\begin{aligned} \sigma_\rho + \sigma_\vartheta &= \frac{2}{3\rho_1}\{p + q - (p - q)\cos 2\alpha\} + 0(1); \\ \sigma_\vartheta - \sigma_\rho + 2i\tau_{\rho\vartheta} &= \frac{2}{3\rho_1}\{i(p - q)\sin 2\alpha\} + 0(1), \end{aligned} \tag{IV.38}$$

where 0 (1) is the bounded part of the component of the stress tensor at $\rho_1 \to 0$.

Using formulas (IV.34) and (IV.38), and also noticing that in plane z on line $y = 0$ ($\vartheta = 0$) the equalities $\sigma_\rho = \sigma_x$, $\sigma_\vartheta = \sigma_y$ and $\tau_{\rho\vartheta} = \tau_{xy}$ hold, we finally obtain

$$\begin{gathered}\sigma_x = \sigma_y = \frac{\sqrt{a}}{3\sqrt{x_1}}\{p + q - (p - q)\cos 2\alpha\} + 0(1);\\ \tau_{xy} = \frac{\sqrt{a}}{3\sqrt{x_1}}(p - q)\sin 2\alpha + 0(1),\end{gathered} \quad \text{(IV.39)}$$

where $x_1 = r$ ($r \ll 1$).

Formulas (IV.39) give values of the coefficients of stress intensity in the neighborhood of an angular point (cusp 0_1 on Fig. 50) during plane extension of a plate with a hole in the form of a hypocycloid with three tips only along coordinate line $y = 0$ ($x > a$). Construction of formulas for determination of these coefficients in an arbitrary point of region D on the basis of equations (IV.24)-(IV.27) involves major calculations. Such formulas can be constructed comparatively simply, if we note the following.

The examined angular points O_j ($j = 1, 2, 3, \ldots$) on the boundary of an infinite plate with a hole (see Figs. 50 and 51) are cusps. In a small neighborhood of such points, as is shown above (see also work [154, 219]), during plane extension-compression of a plate with holes components (σ_r, σ_β, $\tau_{r\beta}$) of the stress tensor in polar system of coordinates (r, β) with its origin in angular point O_j are determined by the following relationships:

$$\begin{gathered}\sigma_r = \frac{1}{4\sqrt{2r}}\Big\{k_{1,j}\Big(5\cos\frac{\beta}{2} - \cos\frac{3}{2}\beta\Big) +\\ + k_{2,j}\Big(-5\sin\frac{\beta}{2} + 3\sin\frac{3}{2}\beta\Big)\Big\} + 0(1);\\ \sigma_\beta = \frac{1}{4\sqrt{2r}}\Big\{k_{1,j}\Big(3\cos\frac{\beta}{2} + \cos\frac{3}{2}\beta\Big) -\\ - 3k_{2,j}\Big(\sin\frac{\beta}{2} + \sin\frac{3}{2}\beta\Big)\Big\} + 0(1);\\ \tau_{r\beta} = \frac{1}{4\sqrt{2r}}\Big\{k_{1,j}\Big(\sin\frac{\beta}{2} + \sin\frac{3}{2}\beta\Big) +\\ + k_{2,j}\Big(\cos\frac{\beta}{2} + 3\cos\frac{3}{2}\beta\Big)\Big\} + 0(1),\end{gathered} \quad \text{(IV.40)}$$

where $k_{1,j}$, and $k_{2,j}$ $(j = 1, 2, 3...)$ are coefficients of the intensity of elastic stresses in the neighborhood of an angular point. According to formula (III.103):

$$k_{1,j}\sqrt{\frac{2}{r}}\cos\frac{\beta}{2} - k_{2,j}\sqrt{\frac{2}{r}}\sin\frac{\beta}{2} + 0(1) = 4\,\mathrm{Re}\,\Phi_j(z_j). \qquad \text{(IV.41)}$$

Here $\Phi_j(z_j)$ is function $\Phi(z)$, referred to local system of polar coordinates (r, β), where $z_j = re^{i\beta}$.

Functions $\Phi_j(z_j)$ for small regions D in the neighborhood of angular points z_{0j} (see Fig. 50) are easy to construct. These functions in accordance with formula (III.96) can be written as:

$$\Phi_j(z_j) = \Phi(z_{0j} + z_j e_i^{i\vartheta}) = \frac{\varphi'(\sigma_j + \zeta_j\sigma_j)}{\omega'(\sigma_j + \zeta_j\sigma_j)},$$

where $j = 1, 2, 3$; $z_j = re^{i\beta}$; $\zeta_j = \rho_1 e^{i\lambda}$ $(r \ll a,\ \rho_1 \ll 1)$; z_j and ζ_j are interconnected by formula (IV.34); $\sigma_j = e^{i\vartheta_j}$ is determined by equalities (IV.31).

Hence and on the basis of formulas (IV.27), (IV.28) and (IV.34) we find

$$\Phi_j(r, \beta) = \frac{1}{3}(2\Gamma + e^{i\vartheta_j}\overline{\Gamma'})\sqrt{\frac{a}{r}}e^{-i\frac{\beta}{2}} + 0(1) \qquad \text{(IV.42)}$$
$$(j = 1, 2, 3),$$

or, taking into account equalities (IV.5a), we have

$$\Phi_j(r, \beta) = \frac{1}{6}\sqrt{\frac{a}{r}}[p + q - (p - q)e^{i(2\alpha+\vartheta_j)}]e^{-i\frac{\beta}{2}} + 0(1),$$

where $j = 1, 2, 3...$

Placing the value of function $\Phi_j(r, \beta)$ in equation (IV.41) and comparing coefficients $k_{1,j}$ and $k_{2,j}$ considering identical harmonics, we obtain

$$\left.\begin{aligned} k_{1,j} &= \frac{\sqrt{2a}}{3}[p + q - (p - q)\cos(2\alpha + \vartheta_j)];\\ k_{2,j} &= \frac{\sqrt{2a}}{3}(p - q)\sin(2\alpha + \vartheta_j), \end{aligned}\right\} \qquad \text{(IV.43)}$$

where values of $\theta_j\,(j=1,\,2,\,3)$ are determined by equalities (IV.31).

It is easy to verify that at $\beta=0$ from formulas (IV.40) and (IV.43) follow (as a special case) formulas (IV.39). Considering in formulas (IV.43) $q=0$, we will obtain the values of coefficient $k_{1,j}$ and $k_{2,j}$, established by other means in work [40].

<u>Case of a hole in the form of an astroid</u> (see Fig. 51). Analogously to the preceding we will determine values of the coefficients of stress intensity in the neighborhood of angular points of the contour of an infinite plate weakened by a hole in the form of an astroid and subjected to biaxial extension. In this case on the basis of formulas (IV.3) and (IV.22) stress function $\Phi(z)$ can be written so:

$$\Phi(z)=\frac{\varphi'(\zeta)}{\omega'(\zeta)}=\frac{1}{\zeta^4-1}\left[\Gamma\zeta^4+\Gamma+\bar{\Gamma}'\zeta^2\right]-\frac{4\bar{a}_1\zeta^2}{9(\zeta^4-1)a}, \qquad \text{(IV.44)}$$

where the connection between variables ζ and z is carried out by the formula

$$z=\omega(\zeta)=\frac{3a}{4}\left(\zeta+\frac{1}{3\zeta^3}\right), \qquad \text{(IV.45)}$$

and constant $\bar{a}_1$ is determined by equality (IV.23) of the preceding section, which for the examined case leads to the expression

$$a_1=\frac{9a}{32}(p-q)(2\cos 2\alpha+i\sin 2\alpha). \qquad \text{(IV.45a)}$$

Tips $O_j\,(j=1,2,3,4)$ of the examined hole in plane z (see Fig. 51) have the following coordinates z_{0j} $(j=1,2,3,4)$: $z_{01}=a;\ z_{02}=ia;\ z_{03}=-a;\ z_{04}=-ia.$ Upon conformal mapping (IV.45) these points pass into points of unit circle $\zeta=\xi+i\eta$, i.e.,

$$\sigma_j=e^{i\theta_j} \qquad (j=1,2,3,4), \qquad \text{(IV.46)}$$

where

$$\theta_1=0;\quad \theta_2=\frac{\pi}{2};\quad \theta_3=\pi;\quad \theta_4=-\frac{\pi}{2}.$$

If we introduce local system of rectangular coordinates $x_jO_jy_j$ $(j = 1, 2, 3, 4)$ with origin in an angular point, then we can write

$$z = z_{0j} + z_j e^{i\Theta_j} \quad (z_j = re^{i\beta});$$
$$\zeta = \sigma_j + \zeta_j e^{i\Theta_j} \quad (\zeta_j = \rho_j e^{i\lambda}). \tag{IV.47}$$

For a small region of the plate in the neighborhood of angular point z_{0j} (see Fig. 51) on the basis of expressions (IV.45) and (IV.47), we find accurate to small $O(\rho_1^3)$ the following relationship between variables z_j and ζ_j [see also formula (IV.34a)]:

$$\zeta_j = \sqrt{\frac{2z_j}{3a}}. \tag{IV.48}$$

Putting expression (IV.47) in (IV.44) and taking into account equalities (IV.48), we obtain

$$\Phi_j(z_j) = \Phi(z_{0j} + z_j e^{i\Theta_j}) = \frac{\sqrt{3a}}{8\sqrt{2r}} e^{-i\frac{\beta}{2}} \Big\{p + q - \\ - \frac{3}{4}(p - q) e^{2i\Theta_j} (2\cos 2\alpha + i \sin 2\alpha)\Big\} + 0(1). \tag{IV.49}$$

Hence and on the basis of equalities (IV.46) we find

$$4\,\mathrm{Re}\,\Phi_j(z_j) = \frac{\sqrt{3a}}{2\sqrt{2r}} \Big\{\Big[p + q - \frac{3}{4}(p - q)\, 2\cos 2\Theta_j \cos 2\alpha\Big] \cos\frac{\beta}{2} - \\ - \frac{3}{4}(p - q)\cos 2\Theta_j \sin 2\alpha \sin\frac{\beta}{2}\Big\} + 0(1).$$

Using this expression and relationship (IV.41) it is easy to obtain formulas for determination of the coefficients of stress intensity in the neighborhood of points O_j $(j = 1, 2, 3, 4)$ during plane extension of a plate with a hole in the form of an astroid (see Fig. 51):

$$\left.\begin{aligned} k_{1,j} &= \frac{\sqrt{3a}}{4}\Big\{p + q - \frac{3}{2}(p - q)\cos 2\Theta_j \cos 2\alpha\Big\}; \\ k_{2,j} &= \frac{3\sqrt{3a}}{16}(p - q)\cos 2\Theta_j \sin 2\alpha, \end{aligned}\right\} \tag{IV.50}$$

where $j = 1, 2, 3, 4$; $\Theta_1 = 0$; $\Theta_2 = \frac{\pi}{2}$; $\Theta_3 = \pi$; $\Theta_4 = -\frac{\pi}{2}$.

Consequently, the principal values of components $(\sigma_r, \sigma_\beta, \tau_{r\beta})$ of the stress tensor in the neighborhood of angular points O_j $(j = 1, 2, 3, 4)$ for this example are determined on the basis of formulas (IV.40) and (IV.50).

4. Calculation of Limit Stresses

According to results of Section 3 Chap. III, and also in accordance with formulas (IV.40) for determination of limit stresses $p_*, q_*\left(\frac{q}{p} = \eta_0\right)$ during plane extension of a plate weakened by a hypocycloidal hole with angular cusps we have the following equations:

$$\begin{gathered} p_* = \min\{p_{*j}\}, \quad q_* = \eta_0 p_* \quad \left(\eta_0 = \frac{q}{p}\right); \\ \lim_{r\to 0}\{\sqrt{r}\,\sigma_\beta(r, \beta_*, p_{*j}, \eta_0)\} = \frac{K}{\pi}. \end{gathered} \tag{IV.51}$$

Here $j = 1, 2, 3\ldots$ is the number of the angular point;

$$\beta_* = 2\,\mathrm{arctg}\,\frac{1 \pm \sqrt{1 + 8n_j^2}}{n_j}, \tag{IV.51a}$$

where the «+» corresponds to values of $k_{1,j} < 0$, and the «—» to values of $k_{1,j} > 0$; parameter $n_j = k_{2,j}/k_{1,j}$.

Case of hypocycloidal hole with three tips. Using formulas (IV.43), (IV.50) and equations (IV.51), one can determine the limit stresses of plates shown on Figs. 50 and 51.

For a plate with a hole in the form of a hypocycloid with three tips (Fig. 50) we have

$$\left.\begin{aligned} p_* &= \min\{p_{*,1}, p_{*,2}, p_{*,3}\}; \quad q_* = \eta_0 p_*; \\ p_{*,j} &= R_1 f_j(\alpha, \eta_0, \beta_*, \vartheta_j); \quad q_{*,j} = \eta_0 p_{*,j}. \end{aligned}\right\} \tag{IV.52}$$

Here $j = 1, 2, 3$; $\eta_0 = \frac{q}{p}$; $R_1 = \frac{\sqrt{2K}}{\pi\sqrt{a}}$; $f_j(\alpha, \eta_0, \beta_*, \vartheta_j) = 6\sqrt{2}\Big\{[1 + \eta_0 - (1 - \eta_0)\cos(2\alpha +$

$$+\,\vartheta_j)]\left(3\cos\frac{\beta_*}{2} + \cos\frac{3}{2}\beta_*\right) - 3(1 - \eta_0)\sin(2\alpha + \vartheta_j)\left(\sin\frac{\beta_*}{2} + \sin\frac{3}{2}\beta_*\right)\Big\}^{-1}; \tag{IV.53}$$

angle β_* is determined by formulas (IV.51a) if in these formulas we substitute the value

$$n_j = \frac{k_{2,j}}{k_{1,j}} = \frac{(1-\eta_0)\sin(2\alpha+\vartheta_j)}{1+\eta_0-(1-\eta_0)\cos(2\alpha+\vartheta_j)}, \tag{IV.54}$$

where

$$j = 1, 2, 3; \quad \vartheta_1 = 0; \quad \vartheta_2 = \frac{2}{3}\pi; \quad \vartheta_3 = -\frac{2}{3}\pi.$$

In the special case when a plate is weakened by a hypocycloidal hole with three tips and is subjected to uniform extension $(p = q, \eta_0 = 1)$, from formulas (IV.52)-(IV.54) we find that local destruction of the plate sets in simultaneously in all angular points $O_j\,(j = 1, 2, 3)$, if external stresses reach

$$p_* = \frac{3}{2\sqrt{2}} \cdot \frac{K\sqrt{2}}{\pi\sqrt{a}} \approx 1{,}06\frac{K\sqrt{2}}{\pi\sqrt{a}}. \tag{IV.55}$$

In this case $\beta_* = 0$; $p_{*,1} = p_{*,2} = p_{*,3} = p_*$.

When the examined plate (Fig. 50) is subjected to uniaxial extension by stressed p $(q = 0, \eta_0 = 0)$, then on the basis of formulas (IV.52)-(IV.54) we have

$$\begin{aligned} &p_* = \min\{p_{*,j}\} \qquad (j = 1, 2, 3); \\ &p_{*j} = R_1 f_j(\alpha, 0, \beta_*, \vartheta_l) \qquad (q_{*j} = 0), \end{aligned} \tag{IV.56}$$

where
$$f_j(\alpha, 0, \beta_*, \vartheta_j) = 6\sqrt{2}\Big\{2\sin(\alpha-\vartheta_j)\Big[3\sin\Big(\alpha-\vartheta_j-\frac{\beta_*}{2}\Big) + \sin\Big(\alpha-\vartheta_j-\frac{3}{2}\beta_*\Big) - 2\cos(\alpha-\vartheta_j)\sin\frac{3}{2}\beta_*\Big]\Big\}^{-1},$$

and angle β_* is determined by formulas (IV.51) if in these formulas we set $\eta_0 = 0$, $n_j = \operatorname{ctg}(\alpha - \vartheta_j)$.

In particular, at $\alpha = \frac{\pi}{2}$ from formulas (IV.56) we find that local destruction of the plate sets in at the neighborhood of point O_1 (see Fig. 50), when external stresses p reach

$$p_{*,1} = \frac{3}{2\sqrt{2}}\frac{K\sqrt{2}}{\pi\sqrt{a}} \qquad (\beta_* = 0).$$

It is easy to note that this formula agrees with formula (IV.55). Values of stresses $p_{*,2}$ and $p_{*,3}$ at $\alpha = \frac{\pi}{2}$ exceed $p_{*,1}$.)

Let us examine now a body in whose structure is a network of defects in the form of hypocycloidal (peaked) cavities (Fig. 53) and assume that these defects are variously oriented and are dispersed all over the volume of the body. Furthermore, we assume that the characteristic linear dimension a of the hypocycloidal cavities (defects) is the mean radius of the circles which can be inscribed around these defects. In this case $R_1 = \frac{\sqrt{2}\,K}{\pi\sqrt{a}}$ is constant for a given material under preassigned conditions (temperature, structure and chemical composition of material). Using formulas (IV.51)-(IV.54), it is possible to construct a diagram of the limit stresses in the case of biaxial extension-compression of such a body. We find for every tip of this cavity the minimum value of function $f_j(\alpha, \eta_0, \beta_*, \vartheta_j)$ for a given value of parameter η_0, when a takes values in the interval $0 \leqslant \alpha \leqslant \frac{\pi}{2}$. Having minimum values of functions $f_j(\alpha, \eta_0, \beta_*, \vartheta_j)$ and using formulas (IV.52), we will determine the limit values of external stresses $p = p_*$ and $q = q_*$ for the examined body.

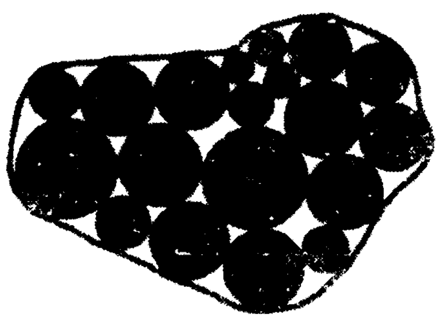


Fig. 53.

On **Figs.** 54 and 55 the change of function $f_j(\alpha, \eta_0, \beta_*, \vartheta_j)$, when $\vartheta_1 = 0$ and $\vartheta_2 = \frac{2}{3}\pi$, and angle α takes values in the interval $0 \leqslant \alpha \leqslant \frac{\pi}{2}$ is graphed. On the basis of such graphs one can determine minimum values

of function $f_j(\alpha, \eta_0, \beta_*, \vartheta_j)$ and the corresponding values of angles $\alpha = \alpha_*$ and $\beta = \beta_*$. These values for $\vartheta_1 = 0$ and fixed values of η_0 are given in Table 15. According to values of α_*, β_* and formulas (IV.52) for our problem (Fig. 50) it is possible to calculate limit stresses $p_{*j}^{(min)}$ and $q_{*j}^{(min)}$ (Table 15).

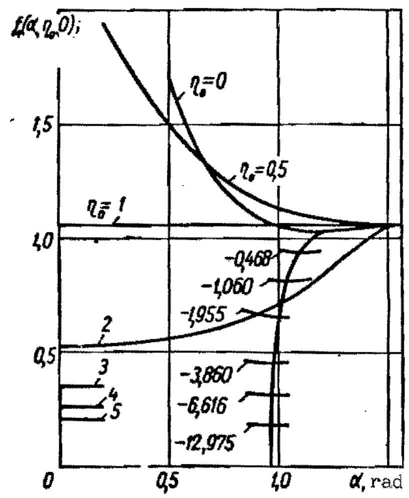


Fig. 54

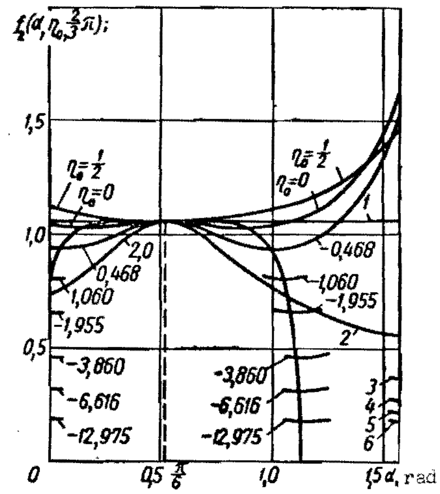


Fig. 55

Table 15.

η_0	α_*, rad	β_*, rad	$\frac{p_*^{min}}{R_1}$	$\frac{q_*^{min}}{R_1}$
∞	0,385	0,622	0,000	1,061
15,0000	0	0,000	0,071	1,061
14,0000	0	0,000	0,076	1,061
12,0000	0	0,000	0,088	1,061
10,0000	0	0,000	0,106	1,061
8,0000	0	0,000	0,133	1,061
5,0000	0	0,000	0,212	1,061
3,0000	0	0,000	0,354	1,061
2,0000	0	0,000	0,530	1,061
1,0000	—	0,000	1,061	1,061
0,5000	1,571	0,000	1,061	0,530
0,0000	1,186	—0,622	1,028	0,000
—0,4680	1,087	—0,894	0,931	—0,436
—1,0602	1,041	—1,062	0,804	—0,852
—1,9555	1,019	—1,187	0,650	—1,271
—3,8603	0,988	—1,303	0,454	—1,753
—6,6160	0,978	—1,326	0,313	—2,069
—12,9750	0,973	—1,410	0,182	—2,355
— ∞	0,966	—1,466	0,000	—2,741

According to Table 15 in coordinate plane qOp the diagram of limit stresses p_*, q_* is built for a solid weakened by hypocycloidal cavities (see Fig. 53) and subjected to biaxial extension-compression

by monotonically increasing external stresses p and $q\left(\frac{q}{p}=\eta_0\right)$. This diagram (Fig. 56) is the same type as diagram 1 on Fig. 43.

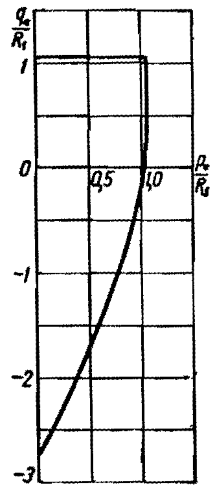


Fig. 56

Let us note that on the basis of **Tables** 13 and 15 we have the following equality:

$$\frac{p_*}{R_1}\approx 1{,}061\,\frac{p_*}{R}\,.$$

This equality can be obtained directly from analysis of formula (IV.56), and also formulas (III.64) and (III.65).

<u>The case of a hypocycloidal hole with four tips</u>. For a body with a hole in the form of an astroid (hypocycloidal cavity with four tips, Fig. 51) on the basis of equations (IV.51) and (IV.51a) and formulas (IV.50) we find

$$\begin{gathered} p_*=\min\{p_{*,1},\ p_{*,2},\ p_{*,3},\ p_{*,4}\};\quad q_*=\eta_0 p_*;\\ p_{*j}=R_1F_j(\alpha,\eta_0,\beta_*,\Theta_j)\ \left(R_1=\frac{\sqrt{2K}}{\pi\sqrt{a}}\right). \end{gathered} \tag{IV.57}$$

Here

$$\begin{gathered} F_j(\alpha,\eta_0,\beta_*,\Theta_j)=\frac{16}{\sqrt{3}}\left\{\left[1+\eta_0-\frac{3}{2}(1-\eta_0)\cos 2\Theta_j\cos 2\alpha\right]\times\right.\\ \times\left(3\cos\frac{\beta_*}{2}+\cos\frac{3}{2}\beta_*\right)-\frac{9}{4}(1-\eta_0)\cos 2\Theta_j\sin 2\alpha\times\\ \left.\times\left(\sin\frac{\beta_*}{2}+\sin\frac{3}{2}\beta_*\right)\right\}^{-1}, \end{gathered} \tag{IV.58}$$

where $j = 1, 2, 3, 4$; $\eta_0 = \frac{q}{p}$; $2a$ is the distance between opposite tips of the astroid; Θ_j — angle equal to $0, \frac{\pi}{2}, \pi$ and $-\frac{\pi}{2}$ respectively for points O_1, O_2, O_3 and O_4 (see Fig. 51); β_* is determined by formulas (IV.51), if in these formulas we substitute the values of coefficients $k_{1,j}, k_{2,j}$, represented by equalities (IV.50); α — angle determining orientation of hole with respect to line of action of stresses p.

In special case, when $q = 0$, $p \neq 0$ $(\eta = 0)$ and $\alpha = \frac{\pi}{2}$, i.e., when a plate with a hole in the form of an astroid is subjected to uniaxial extension by stresses p, directed perpendicularly to the horizontal axis of the astroid, from formulas (IV.57) and (IV.58) we find

$$p_* = p_{*.1} = R_1 F_1\left(\frac{\pi}{2}, 0, 0, 0\right) \qquad (\beta_{*.1} = \beta_{*.3} = 0,\ p_{*.1} = p_{*.3})$$

or, calculating the value of function $F_1\left(\frac{\pi}{2}, 0, 0, 0\right)$, we have

$$p_* = \frac{1{,}6}{\sqrt{3}} R_1 \approx 0{,}92 \frac{\sqrt{2}K}{\pi\sqrt{a}}. \tag{IV.58a}$$

It is helpful to compare this formuls with the formula of Griffith

$$p_*^{(\Gamma)} = \frac{\sqrt{2}K}{\pi\sqrt{l}} \qquad (l = a),$$

assuming that distance $2a$ between the tips of the astroid equals the length $2l$ of an isolated linear macrocrack.

Such comparison shows that the strength of a plate with a hole in the form of an astroid during uniaxial extension is somewhat less than the strength of a plate weakened by a linear crack of corresponding length, i.e., in this case the influence of the size of the hole on a change of the value of limit stresses is immaterial.

On the basis of formulas (IV.57) and (IV.58) and analogously to the above, it is possible to construct a diagram of limit stresses for a body weakened by astroidal holes and subjected to biaxial extension-compression (see work [137]). In works [27, 56, 138, 139] solutions

to problems about maximum equilibrium of a plate weakened by a hole (opened crack) with one or two cusps on its contour also are given.

5. Approximate Determination of Limit Stresses for a Plate Weakened by Circular Holes and Cracks Spreading Outwards on its Contour

Maximum equilibrium of a plate weakened by a circular hole and radial cracks spreading outwards on its contour was first examined by Bowie in work [177]. This work gives an approximate (numerical) resolution of the problem for the case of one and two equal cracks. The exact solution of such problems is difficult. Below an approximate solution to this problem, expounded in work [119], is given.

Cases of two unequal cracks. Let us assume that in an infinite plate (elastic plane xOy, Fig. 57) is a circular hole of radius R and two macrocracks of length l_1 and l_2 $(l_2 \leqslant l_1)$, located on the continuation of the diameter of the hole and spreading outwards on its contour.

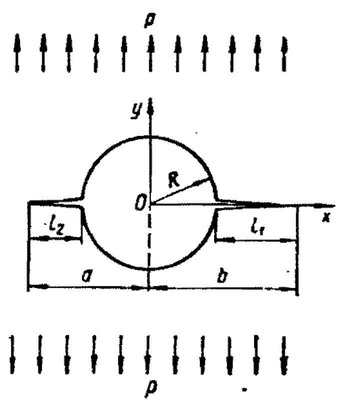


Fig. 57

We assume that the thickness of the plate is equal to unity and cracks are located on segments $-a \leqslant x \leqslant -R$ and $R \leqslant x \leqslant b$, where $b = R + l_1$, $a = R + l_2$. Let us assume further that the plate is stretched by a system of external stresses, symmetric with respect to the plane of location of the cracks (for example, in points at infinity of the plate monotonically increasing stresses $\sigma_y = p$ are applied). Let us determine the limit value of stresses $p = p_*$.

We find the intensity of the rupture elastic stresses $\sigma_y(x, 0)$ in the neighborhood of the ends of the crack. For our problem exact determination of these stresses is difficult. Because of this we will present elastic stresses $\sigma_y(x, 0)$ approximately in the form of the sum

$$\sigma_y(x, 0) \simeq \sigma_y^{(0)}(x, 0) + \sigma_y^{(1)}(x, 0), \tag{IV.59}$$

where $x \leqslant -a$ or $x \geqslant b$; $\sigma_y^{(0)}(x, 0)$ are the elastic tensile stresses appearing in an elastic plate with a circular hole, when such a plate is stretched at infinity by stresses $\sigma_y = p$; $\sigma_y^{(1)}(x, 0)$ are the elastic tensile stresses which appear in an elastic plane with a linear cut along axis Ox at $-a \leqslant x \leqslant b$, when the sides of this cut on sections $-a \leqslant x \leqslant -R$ and $R \leqslant x \leqslant b$ receive normal pressure

$$p_n(x) = \sigma_y^{(0)}(x, 0). \tag{IV.60}$$

Presentation of stresses $\sigma_y(x, 0)$ in the form of a sum (IV.59) is fully justified if $R \ll l_2 \leqslant l_1$. If the radius of the hole is commensurate with the length of the crack, sum (IV.59) gives only a certain approximate value of stresses $\sigma_y(x, 0)$ for our problem.[2]

Stresses $\sigma_y^{(0)}(x, 0)$, in equality (IV.59) are easily calculated for an arbitrary load [103], therefore in the future we will consider that they are known. Stresses $\sigma_y^{(1)}(x, 0)$ can be determined according to results of work [70] by the formula:

$$\sigma_y^{(1)}(x, 0) = \frac{1}{\pi \sqrt{(x-b)(x+a)}} \int_{-a}^{b} \frac{p_n(\xi) \sqrt{(b-\xi)(a+\xi)}}{|x-\xi|} d\xi, \tag{IV.61}$$

where $x \leqslant -a$, $x \geqslant b$, $a \leqslant b$;

$$p_n(\xi) = \begin{cases} \sigma_y^{(0)}(\xi, 0) & \text{when } -a \leqslant \xi \leqslant -R; \\ 0 & \text{when } -R < \xi < R; \\ \sigma_y^{0}(\xi, 0) & \text{when } \quad R \leqslant \xi \leqslant b. \end{cases} \tag{IV.62}$$

Thus, in accordance with results mentioned above and on the basis of formulas (IV.59)-(IV.62) for approximate determination of the value of limit stresses $p = p_*$ we obtain the following equation:

$$\lim_{x \to x_j \pm 0} \left\{ \pm \sqrt{|x - x_j|} \left[\sigma_y^{(0)}(x, 0) + \frac{1}{\pi \sqrt{(x-b)(x+a)}} \times \right.\right.$$
$$\left.\left. \times \int_{-a}^{b} \frac{p_n^*(\xi) \sqrt{(b-\xi)(a+\xi)}}{|x-\xi|} d\xi \right] \right\} = \frac{K}{\pi}, \qquad \text{(IV.63)}$$

where the value of $p_n^*(\xi)$ is determined by formula (IV.62), when $p = p_*$; x_j is the abscissa of one end of the cracks (a, b on Fig. 57).

Stresses $\sigma_y^{(0)}(x, 0)$ do not depend on parameters characterizing the dimensions of the cracks, i.e., on abscissas a and b, therefore equation (IV.64) is easily transformed to the form

$$\lim_{x \to x_i \pm 0} \left\{ \frac{\pm \sqrt{|x - x_i|}}{\sqrt{(x-b)(x+a)}} \int_{-a}^{b} \frac{p_n^*(\xi) \sqrt{(b-\xi)(a+\xi)}}{|x-\xi|} d\xi \right\} = K. \qquad \text{(IV.64)}$$

Hence to determine the limit values of external stresses $p = p_*^b$, after which crack propagation in the direction of abscissa b begins, we have the equality

$$\frac{1}{\sqrt{b+a}} \int_{-a}^{b} \frac{p_n^*(\xi) \sqrt{(b-\xi)(a+\xi)}}{b-\xi} d\xi = K, \qquad \text{(IV.65)}$$

where $p_n^*(\xi)$ is determined by formula (IV.62) when $p = p_*^{(b)}$.

In an analogous way from equation (IV.66) an equation can be obtained to determine stresses $p = p_*^{(a)}$, after which crack propagation begins in the direction of abscissa a. But, inasmuch as for problem $a \ll b$ $(l_2 \ll l_1)$, obviously, $p_*^{(b)} \ll p_*^{(a)}$. Therefore we will limit ourselves to determination of only stresses $p_* = p_*^{(b)}$.

Let us examine more specifically the following problem.

I. The case when in points at infinity of a plate weakened by a circular hole with radial cracks l_1 and l_2 (see Fig. 57) there act tensile stresses $\sigma_y = p$, $\sigma_x = 0$, and $\tau_{xy} = 0$.

II. The case when in points at infinity of such a plate here act stresses $\sigma_y = q, \sigma_x = q,$ and $\tau_{xy} = 0$ (uniform extension), and the contour of the circular hole in problems I and II is free from external stresses.

For these problems stresses $\sigma_y^{(0)}(x, 0)$ have the following form [103]:

$$\sigma_y^{(0)}(x, 0) = p\left(1 + \frac{1}{2}\frac{R^2}{x^2} + \frac{3}{2}\cdot\frac{R^4}{x^4}\right); \tag{IV.66}$$

$$\sigma_y^{(0)}(x, 0) = q\left(1 + \frac{R^2}{x^2}\right). \tag{IV.67}$$

On the basis of formulas (IV.62) and (IV.65)-(IV.67) for determination of limit stresses $p = p_*$ and $q = q_*$ we obtain the following formulas:

$$p_* = \frac{K\sqrt{b+a}}{f_1(a, b)} \quad \text{and} \quad q_* = \frac{K\sqrt{b+a}}{f_2(a, b)}, \tag{IV.68}$$

where

$$f_1(a, b) = \int_{-a}^{-R}\left(1 + \frac{R^2}{2\xi^2} + \frac{3}{2}\frac{R^4}{\xi^4}\right)\sqrt{\frac{a+\xi}{b-\xi}}\,d\xi + \int_{R}^{b}\left(1 + \frac{R^2}{2\xi^2} + \frac{3}{2}\cdot\frac{R^4}{\xi^4}\right)\sqrt{\frac{a+\xi}{b-\xi}}\,d\xi; \tag{IV.69}$$

$$f_2(a, b) = \int_{-a}^{-R}\left(1 + \frac{R^2}{\xi^2}\right)\sqrt{\frac{a+\xi}{b-\xi}}\,d\xi + \int_{R}^{b}\left(1 + \frac{R^2}{\xi^2}\right)\sqrt{\frac{a+\xi}{b-\xi}}\,d\xi. \tag{IV.70}$$

Calculating integrals (IV.69) and (IV.70) for functions $f_1(a, b)$ and $f_2(a, b)$, we obtain the expressions:

$$f_1(a, b) = A(a, b, R)\sqrt{(a+R)(b-R)} - $$
$$- A(a, b, -R)\sqrt{(a-R)(b+R)} -$$
$$- B(a, b, R)\ln\frac{(\sqrt{ab}+\sqrt{(a-R)(b+R)})^2+R^2}{(\sqrt{ab}+\sqrt{(a+R)(b-R)})^2+R^2} +$$
$$+ \frac{a+b}{2}\left(\pi + \arcsin\frac{a-b-2R}{a+b} - \arcsin\frac{a-b+2R}{a+b}\right), \qquad \text{(IV.71)}$$
$$f_2(a, b) = \left(1+\frac{R}{b}\right)\sqrt{(a+R)(b-R)} - \left(1-\frac{R}{b}\right)\sqrt{(a-R)(b+R)} -$$
$$- \frac{R^2\sqrt{ab}}{2ab^2}(a+b)\ln\frac{(\sqrt{ab}+\sqrt{(a-R)(b+R)})^2+R^2}{(\sqrt{ab}+\sqrt{(a+R)(b-R)})^2+R^2} +$$
$$+ \frac{a+b}{2}\left(\pi + \arcsin\frac{a-b-2R}{a+b} - \arcsin\frac{a-b+2R}{a+b}\right), \qquad \text{(IV.72)}$$

where

$$A(a, b, \pm R) = 1 \pm \frac{R}{b} + \frac{R^2}{b^2}\left(\frac{5}{8} + \frac{1}{8}\frac{b}{a}\right) \pm$$
$$\pm \frac{R^3}{b^3}\left(\frac{15}{16} + \frac{1}{4}\cdot\frac{b}{a} - \frac{3}{16}\cdot\frac{b^2}{a^2}\right);$$
$$B(a, b, R) = \frac{R^2\sqrt{ab}}{32a^3b^4}\{8a^2b^2(a+b) + 3R^2(7a^3 - 3a^2b + 5ab^2 - b^3)\};$$

$$a = R + l_2; \quad b = R + l.$$

The Bowie problem. Using formulas (IV.68), (IV.71) and (IV.72) it is easy to solve problems examined in the Bowie work [177]. Thus, if radial cracks have the same length, i.e., $l_1 = l_2 = l$ and, consequently, $a = b$, on the basis of these formulas we have

$$p_* = \frac{K}{\sqrt{2R(1+\varepsilon)}}\cdot\frac{1}{f_1(\varepsilon)};$$
$$q_* = \frac{K}{\sqrt{2R(1+\varepsilon)}}\cdot\frac{1}{f_2(\varepsilon)}, \qquad \text{(IV.73)}$$

where

$$f_1(\varepsilon) = \frac{\pi}{2} - \operatorname{arctg}\frac{1}{\sqrt{2\varepsilon+\varepsilon^2}} + \frac{\sqrt{2\varepsilon+\varepsilon^2}}{(1+\varepsilon)^4}(2+2\varepsilon+\varepsilon^2), \qquad \text{(IV.74)}$$
$$f_2(\varepsilon) = \frac{\pi}{2} - \operatorname{arctg}\frac{1}{\sqrt{2\varepsilon+\varepsilon^2}} + \frac{\sqrt{2\varepsilon+\varepsilon^2}}{(1+\varepsilon)^2} \quad \left(\varepsilon = \frac{l}{R}\right). \qquad \text{(IV.75)}$$

When an infinite plate weakened by a circular hole and one radial crack (Fig. 58) is extended in points at infinity by stresses $\sigma_y = p$ and $\sigma_x = 0$ or $\sigma_y = q$ and $\sigma_x = q$, and $\tau_{xy} = 0$, the limit values of these stresses

are easily calculated by formulas (IV.68), (IV.71) and (IV.72), if in these formulas we set $l_2 = 0$, and $l_1 \neq 0$. In such a case $(a = R)$ we have

$$p_* = \frac{K\sqrt{1+\lambda}}{\sqrt{R}(1+\varepsilon_1)} \cdot \frac{1}{f_1(\lambda)}; \quad q_* = \frac{K\sqrt{1+\lambda}}{\sqrt{R}(1+\varepsilon_1)} \cdot \frac{1}{f_2(\lambda)}. \qquad \text{(IV.76)}$$

Here

$$f_1(\lambda) = A(\lambda)\sqrt{2\lambda(1-\lambda)} - B(\lambda)\ln\frac{1+\lambda}{(1+\sqrt{2(1-\lambda)})^2+\lambda} + \frac{1+\lambda}{2}\left(\frac{\pi}{2} + \arcsin\frac{1-3\lambda}{1+\lambda}\right); \qquad \text{(IV.77)}$$

$$f_2(\lambda) = \frac{1+\lambda}{2}\left(2\sqrt{2\lambda(1-\lambda)} - \lambda\sqrt{\lambda}\ln\frac{1+\lambda}{(1+\sqrt{2(1-\lambda)})^2+\lambda} + \frac{\pi}{2} + \arcsin\frac{1-3\lambda}{1+\lambda}\right),$$

where

$$A(\lambda) = \frac{1}{16}(16 + 15\lambda + 14\lambda^2 + 15\lambda^3);$$

$$B(\lambda) = \frac{\lambda\sqrt{\lambda}}{32}(5 + 23\lambda - 9\lambda^2 + 21\lambda^3);$$

$$\lambda = \frac{R}{b} = \frac{1}{1+\varepsilon_1}; \quad \varepsilon_1 = \frac{l_1}{R}.$$

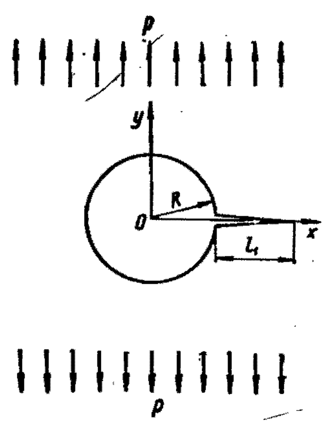


Fig. 58

In accordance with formulas (IV.73)-(IV.77) on Fig. 59 the change of limit loads $\frac{\pi p_* \sqrt{R}}{K}$ and $\frac{\pi q_* \sqrt{R}}{K}$ depending upon relationships $\varepsilon = \frac{l}{R}$ and $\varepsilon_1 = \frac{l_1}{R}$ are graphed (solid lines), where curves 1 correspond

to the case of one radial crack $\left(\varepsilon_1 = \frac{l_1}{R}\right)$, and curves 2 to the case of two radial cracks of identical length $l\left(\varepsilon = \frac{l}{R}\right)$. On these graphs the dotted line represents the dependences of these limit loads, established in work [177]. Comparison of graphs shows that approximate solution by formulas (IV.73)-(IV.77) when $\frac{l}{R} > 0.5$ will agree well with the calculations of Bowie.

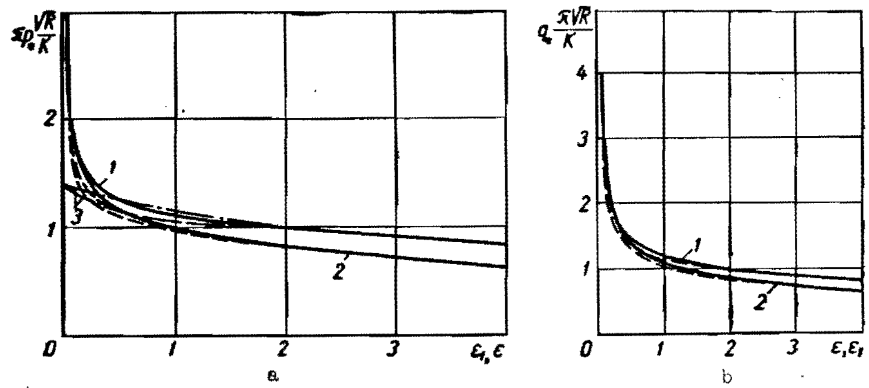


Fig. 59

If the length of the radial cracks is small as compared to the radius of the circular hole, the propagation of such cracks obviously is determined by the size of the rupture stresses a thing directly on the contour of the hole. As is easily seen from formulas (IV.66) and (IV.67), in the case of uniaxial extension the greatest rupture stresses on the contour of the hole are $3p$, and in the case of uniform extension $2q$. It follows from this that for our problems the ratio of limit stresses $\frac{p_*}{q_*} \to \frac{2}{3}$ as $l \to 0$.

Using formulas (IV.73)-(IV.75), we easily find

$$\lim_{l \to 0} \frac{p_*}{q_*} = \lim_{\varepsilon \to 0} \frac{f_2(\varepsilon)}{f_1(\varepsilon)} = \frac{2}{3},$$

i.e., these formulas — in the limit as $l \to 0$ $(\varepsilon \to 0)$ — give the expected result. We have the same result with formulas (IV.75)-(IV.77), when $l_1 \to 0$ $(\varepsilon_1 \to 0)$.

If the length of the radial cracks is great enough $(l \gg R)$ and it is possible to consider that $\varepsilon \to \infty$, $R \to 0$ $(\varepsilon R = l = \mathrm{const})$, then by formulas (IV.73)-(IV.75) or (IV.76)-(IV.77) we obtain the Griffith formulas for an isolated crack.

For comparison on Fig. 59a (curves 3) the change of critical load $\pi p_* \frac{\sqrt{R}}{K}$ is graphed for the case when in the plate there is an isolated linear crack of length $2(R+l)$ or $2\left(R+\frac{1}{2}l\right)$, and in points at infinity of the plate there act external tensile stresses $\sigma_y = p$.

Comparison of curves on Fig. 59a shows that when $l/R > 0.5$ the limit values of stresses $p = p_*$ for a plate weakened by a circular hole and radial cracks radiating on its contour, and for a plate weakened by a linear crack of corresponding length $2(R+l)$ or $2\left(R+\frac{1}{2}l\right)$, are very close, so that the presence of a hole has almost no effect on a change of the limit stress.

In works [21, 53-55, 135] other methods are proposed for approximation of limit stresses for an infinite plate weakened by a curvilinear hole (circular, elliptic, hypocycloidal) and macrocracks radiating outwards on its contour, and concrete examples are examined. In particular, in work [21] the case of a plate weakened by a circular hole, is investigated in detail when on the contour of the hole there emerge n cracks of equal length, and the plate is subjected to uniform extension. It is shown that the least value of limit stresses for such a case is obtained when on the contour of the hole there are three symmetrically located cracks.

The propagation of a linear cracks radiating on the contour of a curvilinear hole in a brittle body when the body is subjected to antiplane deformation investigated in works [11, 152].

6. Limit Load for a Half-Plane with a Crack Radiating on its Lateral Face

The maximum equilibrium of an elastic half-plane with linear crack radiating on its free edge (Fig. 60) was examined in works of Wigglesworth [225], Irwin [199] and Bueckner [179]. Inasmuch as on the contour of the examined region there are angular points different from cusps, then, as was noted above, an exact solution of the corresponding problem of the theory of elasticity for this region is difficult. Therefore it is difficult to construct exact formulas for determination of limit stresses.

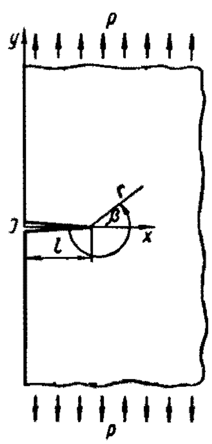


Fig. 60

In works [179, 199, 225] different means are used to construct approximate expressions for determination of the field of elastic stresses in a half-plane in the neighborhood of the end part of the examined crack.[3] These solutions can be used for approximate determination of the limit load of our problem. Thus, on the basis of results of work [225] for determination of elastic stresses near the end of the examined crack we have the following relationships:

$$\sigma_x + \sigma_y = 1{,}586p\sqrt{\frac{l}{r}}\sin\frac{\beta}{2} + 0(1); \tag{IV.78}$$

$$\sigma_x - \sigma_y + 2i\tau_{xy} = -\,0{,}793p\sqrt{\frac{l}{r}}\,e^{\frac{3i\beta}{2}}\sin\beta + 0(1),$$

where β — vectorial angle read counterclockwise from the lower side of the crack; r – polar radius with origin in tip of crack; l – length of crack (see Fig. 60).

On the continuation of the crack, i.e., when $\beta = \pi$, from relationships (IV.78) we find

$$\sigma_y = \sigma_x = 0{,}793 p \sqrt{\frac{l}{r}} + 0(1), \quad \tau_{r\beta} = 0. \tag{IV.78a}$$

Hence and on the basis of equation (1.47) for determination of limit load $p = p_*$ we have the formula

$$p_* \approx 0{,}9 \frac{\sqrt{2}K}{\pi \sqrt{l}}. \tag{IV.79}$$

In work [8] values of limit stresses $p = p_*$ are determined for the examined problem on the basis of results [179, 199]. Comparison of these established that in all cases the obtained approximate values of p_* are close to values represented by formula (IV.79).

Footnotes

[1]In the general case of angular points on the contour bounded by the examined region, when solving the problems of the plane theory of elasticity it is necessary either to smooth the angular points of the contour [150] (if this is possible for the examined problem) and to use the method of N. I. Muskhelishvili, or to use other promising means [17, 143, 144].

[2]Such an approach to determination of stresses $\sigma_y(x,0)$ can be examined as a certain analogy of the method of successive approximations, proposed in works [95, 172].

[3]Work [112] examines the theory of elasticity for a half-plane with a cut (see Fig. 60), when the half-plane is subjected to uniaxial extension. In this case of components of the stress tenser are represented in the form of quadratures of the function which is the solution of a certain integral equation. But it is not possible to construct the solution of this equation in closed form, therefore practically the problem reduces to approximate consideration.

C H A P T E R V

BEND OF STRIPS (BEAMS) WEAKENED BY LINEAR CRACKS

1. General Remarks

Below we examine certain problems about maximum equilibrium of strips (beams) weakened by through linear macroscopic cracks, when the strip is bent in its plane by an assigned system of external loads, for example, constant bending moments evenly distributed pressure or concentrated force. The course of solving of such problems, as in the preceding chapter is as follows: in the beginning a corresponding problem of the theory of elasticity is solved and rupture stresses in the neighborhood of the ends of the crack are determined, and then on the basis of equations (III.34) and (III.35) formulas are constructed or determination of the limit load. Results in this chapter have been basically published in works [86, 87, 122, 124].

2. Determination of Stresses in the Neighborhood of a Crack During the Bending of a Strip

Let us examine an elastic isotropic strip (beam) weakened by a through crack directed perpendicularly to the lateral faces of the strip (Fig. 61). Let us designate by 2h and 2τ respectively the width and thickness of the strip, and by $2l$ the length of the crack. Let us introduce rectangular system of cartesian coordinates xOy and assume that in plane xOy the crack is located along the x-axis when $-a \leq x \leq b$, where a, b are the abscissas of the ends of the crack ($2l = b + 2$).

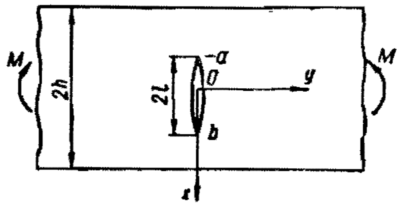


Fig 61.

Let us assume that on this strip there act external loads (bending moments, evenly distributed along the length of the strip by pressure or concentrated force), located in the middle plane of the strip; the edges of the strip, parallel to planes xOy, are assumed free from external stresses. Under the impact of external loading in the zone of compression stresses the sides of the crack on certain section $\lambda_1 \leq x \leq \lambda_2$, where $-a \leq \lambda_1$ and $\lambda_2 \leq b$, will come into contact, which will lead to the appearance of contact stresses on a given section of the sides of the crack. Outside this section the sides of the crack will be free from contact stresses. Consequently, on the boundary of the region of contact stresses must be limited. Parameters λ_1 and λ_2, determining the boundary of the region of contact between the sides of the crack, must be found in the course of solving the problem.[1]

The problem consists in determination of contact stresses on section (λ_1, λ_2), of the stressed-deformed state outside the crack, and also in determination of the external load after which the crack starts to propagate across the cross section of the strip.

For our problem on the contour of the crack we have the following boundary conditions:

on the contact section, i.e., when $y = 0$, $\lambda_1 \leq x \leq \lambda_2$,

$$\sigma_y^+(x, 0) = \sigma_y^-(x, 0), \quad v^+(x, 0) = v^-(x, 0) = 0, \tag{V.1}$$

where

$$\sigma_y^+(x, 0) = \sigma_y(x, +0); \quad \sigma_y^-(x, 0) = \sigma_y(x, -0);$$
$$v^\pm(x, 0) = v(x, \pm 0);$$

on sections of the sides of the crack free from external stresses,

$$\sigma_y^+(x, 0) = \sigma_y^-(x, 0) = 0. \tag{V.2}$$

Furthermore, we assume that tangential stresses on the sides of the crack obey Coulomb's law, i.e.,

$$\tau_{x,y}^+(x, 0) = \rho_0 \sigma_y^+(x, 0) \text{ when } -a \leqslant x \leqslant b, \tag{V.3}$$

where ρ_0 – coefficient of friction.

As it is known [103] (see also Chapter II), components σ_x, σ_y, τ_{xy} of the stress tensor and components u, v of the displacement vector under conditions of a two-dimensional problem of theorems of elasticity are expressed through two analytic functions $\Phi(z)$ and $\Omega(z)$ by the following formulas:

$$\sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi(z)}] \qquad (z = x + iy); \tag{V.4}$$

$$\sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}; \tag{V.5}$$

$$2G(u' + iv') = \varkappa\Phi(z) - \Omega(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)} \tag{V.6}$$

$$\left(u' = \frac{\partial u}{\partial x}, \quad v' = \frac{\partial v}{\partial x}\right).$$

Functions $\Phi(z)$ and $\Omega(z)$ must be determined on the basis of boundary conditions of the problem. For the above problem we will examine in the beginning two auxiliary functions of the following form:

$$\begin{aligned} \Phi_0(z) &= A_0 z^3 + A_1 z^2 + A_2 z + A_3; \\ \Omega_0(z) &= B_0 z^3 + B_1 z^2 + B_2 z + B_3. \end{aligned} \tag{V.7}$$

These functions depending upon the values of coefficients A_j and B_j (j = 0, 1, 2, 3) determine the state of strain in a strip without cracks. For example, considering in formulas (V.7)

$$\begin{gathered} A_0 = 0; \quad A_1 = 0; \\ A_2 = \frac{M}{4J}; \quad A_3 = 0; \\ B_0 = 0; \quad B_1 = 0; \\ B_2 = \frac{3M}{4J}; \quad B_3 = 0, \end{gathered} \tag{V.8}$$

where J – moment of inertia of strip, $J = 4\tau h^3/3$ and, using relationships (V.4)-(V.5), it is easy to verify that in this case functions $\Phi_0(z)$ and $\Omega_0(z)$ give a solution to the problem about pure bend by M moments of an infinite strip (beam) without a crack (see Fig. 61).

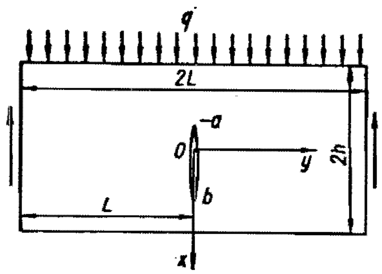


Fig. 62.

Analogously to the preceding, on the basis of formulas (V.4)-(V.5) and (V.7) we will find that when

$$A_0 = \frac{q}{24J};\quad A_1 = 0;\quad A_2 = \frac{q}{8J}\left(L^2 + \frac{3h^2}{5}\right);\quad A_3 = \frac{-qh^3}{12J};$$
$$B_0 = \frac{7q}{24J};\quad B_1 = 0;\quad B_2 = \frac{q}{8J}\left(3L^2 - \frac{11h^2}{5}\right);\quad B_3 = \frac{qh^3}{12J} \tag{V.9}$$

(designations are shown on Fig. 62) functions $\Phi_0(z)$ and $\Omega_0(z)$ give a solution to the problem about the bend of a beam of length 2L without a crack, when the beam is loaded by a uniform pressure of intensity q. Moreover, is assumed that the beam is freely located on two supports, and the support reactions are defined as tangential stresses applied to the ends of the beam. It is not difficult to show also on the basis of formulas (V.4), (V.5) and (V.7) that if

$$A_0 = 0;\quad A_1 = \frac{-iQ}{8J};\quad A_2 = \frac{-Q(2L-d)}{4J};\quad A_3 = 0;$$
$$B_0 = 0;\quad B_1 = \frac{5iQ}{8J};\quad B_2 = \frac{-3Q(2L-d)}{4J};\quad B_3 = \frac{-iQh^2}{2J} \tag{V.10}$$

(designations are shown on Fig. 63), then functions $\Phi_0(z)$ and $\Omega_0(z)$ give a solution to the problem about the bend of a rigidly fixed overhung strip (without a crack) under the action of constant transverse force Q applied on its free end.

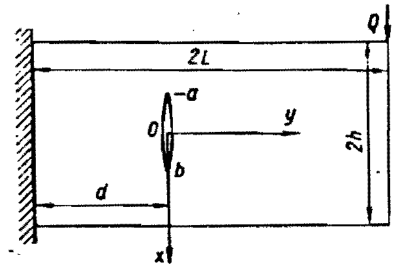


Fig. 63.

The presence in the strip of a stress concentrator in the form of a crack will lead to a disturbance of the field of elastic stresses in the neighborhood of this crack. Far from the crack the stressed-deformed state in the strip for the shown loads is determined by formulas (V.4)-(V.7) if values of coefficients A_j and B_j ($j = 0, 1, 2, 3$) are determined by equalities (V.8)-(V.10), and on the contour of the crack boundary conditions (V.1)-(V.3) hold.

The stressed-deformed state in the neighborhood of the crack is determined approximately in that meaning [103, 150] which will satisfy boundary conditions of the problem on the contour of the crack, i.e., conditions (V.1)-(V.3), and require that a considerable distance from the crack the state of strain in the strip agrees with the state of strain determined by functions (V.7):

$$\lim_{|z|\to\infty} \Phi(z) = \Phi_0(z),$$
$$\lim_{|z|\to\infty} \Omega(z) = \Omega_0(z). \tag{V.11}$$

If in formulas (V.4) and (V.5) we cross to boundary values on the contour of the crack, i.e., set $y \to \pm 0$, and consider boundary conditions (V.1)-(V.3), then we will obtain the following problem of linear coupling of boundary values of the unknown functions $\Phi(z)$ and $\Omega(z)$:

$$[\Phi(t) + \Omega(t)]^+ + [\Phi(t) + \Omega(t)]^- = 2p(t);$$
$$[\Phi(t) - \Omega(t)]^+ - [\Phi(t) - \Omega(t)]^- = 0,$$

where $-a \leq t \leq b$; t — the affix of the point of contour of the crack ($t = x$); $p(t) = (1 - i\rho_0)\sigma_y^+(t, 0)$ — on section of contact ($\lambda_1 \leq t \leq \lambda_2$); $p(t) = 0$ — on nontouching sections of the sides of the crack.

Solving this problem [103] and taking into account behavior of functions $\Phi(z)$ and $\Omega(z)$ at $|z| \to \infty$, we find

$$\Phi(z) = \frac{1}{2\pi i \sqrt{(z+a)(z-b)}} \int_{\lambda_1}^{\lambda_2} \frac{\sqrt{(a+t)(t-b)}}{t-z} p(t)\,dt + \frac{P_n(z)}{\sqrt{(z+a)(z-b)}} + \frac{1}{2}[\Phi_0(z) - \Omega_0(z)]; \tag{V.12}$$

$$\Omega(z) = \frac{1}{2\pi i \sqrt{(z+a)(z-b)}} \int_{\lambda_1}^{\lambda_2} \frac{\sqrt{(a+t)(t-b)}}{t-z} p(t)\,dt + \frac{P_n(z)}{\sqrt{(z+a)(z-b)}} - \frac{1}{2}[\Phi_0(z) - \Omega_0(z)], \tag{V.13}$$

where functions $\Phi_0(z)$ and $\Omega_0(z)$ are determined by equalities (V.7), and polynomial $P_n(z)$ has the form

$$P_n(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_0. \tag{V.14}$$

The power of polynomial (V.14) and its coefficients c_0, c_1, ..., c_n are determined from conditions of the behavior of functions $\Phi(z)$ and $\Omega(z)$ in the neighborhood of point $|z| = \infty$. Functions $\Phi(z)$ and $\Omega(z)$ are holomorphic in the region outside the crack and, according to conditions (V.11), for large $|z|$ have the form

$$\begin{aligned} \Phi(z) &= \Phi_0(z) - \frac{X+iY}{2\pi(1+\varkappa)} \cdot \frac{1}{z} + 0\left(\frac{1}{z^2}\right); \\ \Omega(z) &= \Omega_0(z) + \frac{\varkappa(X+iY)}{2\pi(1+\varkappa)} \cdot \frac{1}{z} + 0\left(\frac{1}{z^2}\right), \end{aligned} \tag{V.15}$$

where X, Y are components of the main vector of external forces applied to the contour of the crack (for the examined problems $X = 0$ and $Y = 0$).

Thus, to determine coefficients c_0, c_1, c_2, ..., c_n it is necessary that function $\Phi(z)$, represented by formula (V.12), be

expanded in a series in powers of z in the neighborhood of point $|z| = \infty$ and that this expansion be compared with expression (V.15). Taking into account formulas (V.7)-(V.13) and carrying out the necessary calculations, coefficients of polynomial (V.14) can be determined thus:

1) considering pure bend of a strip with a crack (designations are given on Fig. 61)

$$c_0 = -\frac{M}{16J}(b+a)^2; \quad c_1 = -\frac{M}{4J}(b-a);$$
$$c_2 = \frac{M}{2J}; \quad c_n = 0 \text{ when } n \geqslant 3; \qquad \text{(V.16)}$$

2) considering the bend of a strip with a crack under the action of an evenly distributed load (for designations see Fig. 62)

$$\left.\begin{aligned}
c_0 &= -\frac{q}{32J}(a+b)^2\left[L^2 - \frac{2}{5}h^2 + \frac{1}{24}(5a^2 - 6ab + 5b^2)\right];\\
c_1 &= \frac{q}{96J}(a-b)\left[(a+b)^2 + 12\left(L^2 - \frac{2}{5}h^2\right)\right];\\
c_2 &= -\frac{q}{48J}\left[(a+b)^2 - 12\left(L^2 - \frac{2}{5}h^2\right)\right];\\
c_3 &= \frac{q}{12J}(a-b); \quad c_4 = \frac{q}{6J}; \quad c_n = 0 \text{ when } n \geqslant 5;
\end{aligned}\right\} \qquad \text{(V.17)}$$

3) considering the bend of a cantilever with a crack (for designations see Fig. 63)

$$\left.\begin{aligned}
c_0 &= -\frac{iQ}{64J}(b-a)[(b+a)^2 - 8h^2] + \frac{Q}{16J}(2L-d)(b+a)^2;\\
c_1 &= -\frac{iQ}{32J}[(b+a)^2 + 8h^2 + 8i(b-a)(2L-d)];\\
c_2 &= -\frac{Q}{8J}\left[4(2L-d) + (b-a)i\right];\\
c_3 &= \frac{iQ}{4J}, \quad c_n = 0 \text{ when } n \geqslant 4.
\end{aligned}\right\} \qquad \text{(V.18)}$$

Determination of contact stresses. For final determination of functions $\Phi(z)$ and $\Omega(z)$ it is still necessary to find contact stresses $\sigma_y^+(t_0)$ $[p(t_0) = (1 - i\rho_0)\sigma_y^+(t_0)]$ on the contact section on the contact section between the edges of the crack, i.e., when $\lambda_1 \leq t_0 \leq \lambda_2$. To do this we examine formula (V.6) and its conjugate

formula. On the basis of these formulas, and carrying out passage to the limit on the contour of the crack when $y \to \pm 0$, we obtain the following contour relationship:

$$4G(v'^{+} - v'^{-}) = \varkappa [\Phi^{+}(t_0) - \Phi^{-}(t_0) + \overline{\Phi^{+}}(t_0) - \overline{\Phi^{-}}(t_0)] + \\ + [\Omega^{+}(t_0) - \Omega^{-}(t_0) + \overline{\Omega^{+}}(t_0) - \overline{\Omega^{-}}(t_0)], \qquad (V.19)$$

where $-a \leq t_0 \leq b$.

To establish the connection between functions $\overline{\Phi}(z)$, $\overline{\Omega}(z)$ and $\Phi(z)$, $\Omega(z)$ on the basis of formula (V.5) and contour conditions (V.1) when $z \to t_0 \pm 0$ we obtain the following problem of linear conjugation [98]:

$$\begin{aligned} &[(1 + i\rho_0)\Phi + (1 - i\rho_0)\overline{\Phi} - (1 + i\rho_0)\Omega - (1 - i\rho_0)\overline{\Omega}]^{+} = \\ &= [(1 + i\rho_0)\Phi + (1 - i\rho_0)\overline{\Phi} - (1 + i\rho_0)\Omega - (1 - i\rho_0)\overline{\Omega}]^{-}; \\ &[(1 + i\rho_0)\Phi - (1 - i\rho_0)\overline{\Phi} + (1 + i\rho_0)\Omega - (1 - i\rho_0)\overline{\Omega}]^{+} = \\ &= -[(1 + i\rho_0)\Phi - (1 - i\rho_0)\overline{\Phi} + (1 + i\rho_0)\Omega - (1 - i\rho_0)\overline{\Omega}]^{-}. \end{aligned} \qquad (V.20)$$

Solving this problem [103], we find

$$\begin{aligned} &(1 + i\rho_0)[\Phi(z) - \Omega(z)] + (1 - i\rho_0)[\overline{\Phi}(z) - \overline{\Omega}(z)] = R_n(z); \\ &(1 + i\rho_0)[\Phi(z) + \Omega(z)] - (1 - i\rho_0)[\overline{\Phi}(z) + \overline{\Omega}(z)] = \\ &= \frac{D_n(z)}{\sqrt{(z + a)(z - b)}}, \end{aligned} \qquad (V.21)$$

where

$R_n(z)$ and $D_n(z)$ —

polynomials to the degree n (for our problems $n \leq 4$),

$$R_n(z) = r_0 + r_1 z + r_2 z^2 + r_3 z^3 + \ldots + r_n z^n,$$
$$D_n(z) = d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \ldots + d_n z^n.$$

Coefficients r_j, d_j, ($j = 0, 1, 2, \ldots, n$) and the degree of these polynomials are determined on the basis of equalities (V.21) and conditions of behavior of functions $\Phi(z)$, $\Omega(z)$ and $\overline{\Phi}(z)$, $\overline{\Omega}(z)$ in the neighborhood of a point at infinity of the strip, i.e., from conditions that when $z \to \infty$ we have

$$\Phi(z) \to \Phi_0(z), \quad \Omega(z) \to \Omega_0(z), \quad \overline{\Phi}(z) \to \overline{\Phi}_0(z), \quad \overline{\Omega}(z) \to \Omega_0(z).$$

In particular, using these relationships and equalities (V.7), (V.15), and (V.21), for determination of coefficients d_j ($j = 0$, 1, 2, 3, 4) of polynomial $D_n(z)$ we have the following expressions[2]:

$$\left.\begin{aligned}
&(1 + i\rho_0)(A_0 + B_0) - (1 - i\rho_0)(\overline{A}_0 + \overline{B}_0) = a_1 d_1;\\
&(1 + i\rho_0)(A_1 + B_1) - (1 - i\rho_0)(\overline{A}_1 + \overline{B}_1) = a_1 d_3 + a_2 d_4;\\
&(1 + i\rho_0)(A_2 + B_2) - (1 - i\rho_0)(\overline{A}_2 + \overline{B}_2) =\\
&\qquad = a_1 d_2 + a_2 d_3 + a_3 d_4;\\
&(1 + i\rho_0)(A_3 + B_3) - (1 - i\rho_0)(\overline{A}_3 + \overline{B}_3) =\\
&\qquad = a_1 d_1 + a_2 d_2 + a_3 d_3 + a_4 d_4;\\
&a_1 d_0 + a_2 d_1 + a_3 d_2 + a_4 d_3 + a_5 d_4 = 0,
\end{aligned}\right\} \tag{V.22}$$

where

$$a_1 = 1; \quad a_2 = \frac{b - a}{2}; \quad a_3 = \frac{2(b^2 + a^2) + (b - a)^2}{8};$$

$$a_4 = \frac{4(b^3 - a^3) + (b - a)^3}{16};$$

$$a_5 = \frac{35a^4 - 20a^3 b + 18a^2 b^2 - 20ab^3 + 35b^4}{128}$$

Hence and on the basis of formulas (V.8)-(V.10) the polynomials for our problems can be represented so:

1. Considering the pure bend of a strip with a crack (see Fig. 61) external loads are symmetric with respect to the plane of location of the crack and, consequently, $\tau_{xy}(x, 0) = 0$. Therefore

$$\rho_0 = 0, \quad D_n(z) = 0. \tag{V.23}$$

2. Considering the bend of a strip with a crack (see Fig. 62) under the effect of an evenly distributed load analogous to point 1

$$\rho_0 = 0, \quad D_n(z) = 0.$$

3. Considering the bend of an arm with a crack (see Fig. 68)

$$\rho_0 \neq 0, \quad D_n(z) = D_3(z) = d_3 z^3 + d_2 z^2 + d_1 z + d_0. \tag{V.23a}$$

where

$$d_0 = -\frac{iQ}{16J}(b-a)[(b+a)^2 - 8h^2] + \frac{i\rho_0 Q}{4J}(2L-d)(b+a)^2;$$
$$d_1 = -\frac{iQ}{8J}[8h^2 - 8\rho_0(2L-d)(b-a) + (b+a)^2];$$
$$d_3 = \frac{iQ}{J}; \quad d_2 = -\frac{iQ}{2J}[4\rho_0(2L-d) + (b-a)].$$

Adding and subtracting relationships (V.21), we find

$$\begin{aligned} \overline{\Phi}(z) &= \frac{1+i\rho_0}{1-i\rho_0}\Omega(z) + \frac{1}{2(1-i\rho_0)}\left[R_n(z) - \frac{D_n(z)}{\sqrt{(z+a)(z-b)}}\right]; \\ \overline{\Omega}(z) &= \frac{1+i\rho_0}{1-i\rho_0}\Phi(z) - \frac{1}{2(1-i\rho_0)}\left[R_n(z) + \frac{D_n(z)}{\sqrt{(z+a)(z-b)}}\right]. \end{aligned} \tag{V.24}$$

By formulas (V.24) it is easy to determine boundary values of functions $\overline{\Phi}(z)$ and $\overline{\Omega}(z)$ on the contour of the crack, i.e., when $y = \pm 0$. If boundary values of $\overline{\Phi}(z)$ and $\overline{\Omega}(z)$ are placed in contour condition (V.19) and we consider that

$$\Phi^+(t_0) - \Phi^-(t_0) = \Omega^+(t_0) - \Omega^-(t_0)$$

(see formula (V.12) and (V.13)), then as a result of simple transformations we obtain

$$2Gi(v'^+ - v'^-) = \frac{\varkappa+1}{1-i\rho_0}\left[\Phi^+(t_0) - \Phi^-(t_0) - \frac{D_n(t_0)}{2\sqrt{(t_0+a)(t_0-b)}}\right], \tag{V.25}$$

where t_0 – affix of the contour of the crack.

Using the Sokhotskiy-Plemel' formulas [103], on the basis of expression (V.12) we find

$$\Phi^+(t_0) - \Phi^-(t_0) = \frac{1}{\sqrt{(t_0+a)(t_0-b)}} \times \left\{\frac{1}{\pi i}\int_{\lambda_1}^{\lambda_2} \frac{\sqrt{(t+a)(t-b)}\,p(t)}{t-t_0}dt + 2P_n(t_0)\right\}.$$

Placing this expression in equation (V.25) and taking into account that according to contour conditions (V.1), the contact section between the sides of the crack, i.e., when $\lambda_1 \leq t_0 \leq \lambda_2$, $v^{1+} = v^{1-} = 0$, we obtain for determination of contact stresses $p(t_0) = (1 - i\rho_0)\sigma_y^+(t_0, 0)$ the following singular integral equation:

$$\frac{1}{\pi i}\int_{\lambda_1}^{\lambda_2} \frac{\sqrt{(t+a)(t-b)}}{t-t_0} p(t)\,dt + 2\left[P_n(t_0) - \frac{1}{4}\bar{D}_n(t_0)\right] = 0, \tag{V.26}$$

or

$$\frac{1}{\pi i}\int_{\lambda_1}^{\lambda_2} \frac{\sqrt{(t+a)(t-b)}}{t-t_0} p(t)\,dt + 2(1-i\rho_0)(s_0 + s_1 t_0 + s_2 t_0^2 + s_3 t_0^3 + s_4 t_0^4) = 0,$$

where $\lambda_1 \leq t_0 \leq \lambda_2$.

In equation (V.26) coefficients s_j ($j = 0, 1, 2, 3, 4$) are expressed by the equalities:

1) considering the bend of a strip with a crack by constant moments M (see Fig. 61) or the bend of a strip by evenly distributed load q (see Fig. 62)

$$s_j = c_j \qquad (j = 0, 1, 2, 3, 4), \tag{V.27}$$

where values of c_j are represented by equalities (V.16) and (V.17);

2) considering the bend of a cantilever with a crack (see Fig. 63)

$$s_j = \mathrm{Re}(c_j), \tag{V.28}$$

where values of c_j are determined by equalities (V.18).

If we consider that

$$p(t_0) = (1 - i\rho_0)\,\sigma_y^+(t_0) \qquad (\lambda_1 \leqslant t_0 \leqslant \lambda_2), \tag{V.29}$$

then integral equation (V.26) can be transformed to the form

$$\frac{1}{\pi i}\int_{\lambda_1}^{\lambda_2} \frac{\sqrt{(t+a)(t-b)}\,\sigma_y^+(t)}{t-t_0}\,dt + 2(s_0 + s_1 t_0 + s_2 t_0^2 + s_3 t_0^3 + s_4 t_0^4) = 0, \tag{V.30}$$

where $\lambda_1 \leq t_0 \leq \lambda_2$; $\sigma_y^+(t_0)$ – unknown contact stresses.

Solving this equation[3] and satisfying conditions of the boundedness of contact stresses when $t_0 = \lambda_1$ and $t_0 = \lambda_2$, we find the necessary formulas for reading stresses $\sigma_y^+(t_0)$.

1. Considering pure bend by constant moments M of a strip with a crack (see Fig. 61) we have

$$\sigma_y^+(t_0) = \frac{M}{2J}(2t_0 + a - b + \lambda_1 + \lambda_2)\sqrt{\frac{(t_0-\lambda_1)(t_0-\lambda_2)}{(t_0+a)(t_0-b)}}, \tag{V.31}$$

where $\lambda_1 \leq t_0 \leq \lambda_2$; values of λ_1 and λ_2 are determined by the equation

$$3\lambda_1^2 + 2\lambda_1\lambda_2 + 3\lambda_2^2 + 2(a-b)(\lambda_1+\lambda_2) - (b+a)^2 = 0. \tag{V.32}$$

Obviously, for our case (see Fig. 61 and also remarks on p. 151) the beginning of the contact region between the sides of the crack is the tip of the crack located in the region of compression stresses. Therefore in this case we can write

$$\lambda_1 = -a. \tag{V.32a}$$

Hence on the basis of equation (V.32) we find

$$\lambda_2 = -\frac{1}{3}b. \tag{V.33}$$

2. Considering the bend of a strip with a crack under the action of evenly distributed load q (see Fig. 62) we obtain

$$\sigma_y^+(t_0) = \left(m_3 t_0^3 + m_2 t_0^2 + m_1 t_0 + m_0\right) \sqrt{\frac{(t_0-\lambda_1)(t_0-\lambda_2)}{(t_0+a)(t_0-b)}}. \qquad (V.34)$$

Here $\lambda_1 \leq t_0 \leq \lambda_2$, where $\lambda_1 = -a$ (analogously to the preceding); λ_2 is determined from the equation

$$\begin{gathered} 35\lambda_2^4 - 20b\lambda_2^3 - 6b^2\lambda_2^2 - 4b^3\lambda_2 - 5b^4 + \\ + 24\left(L^2 - \frac{2}{5}h^2\right)\left(3\lambda_2^2 - 2b\lambda_2 - b^2\right) = 0; \end{gathered} \qquad (V.35)$$

coefficients m_j ($j = 0, 1, 2, 3$) are expressed by the equalities

$$\left.\begin{aligned} m_0 &= \frac{q}{48J}\left\{5\lambda_2^3 - 3b\lambda_2^2 - b^2\lambda_2 - b^3 + 12\left(L^2 - \frac{2}{5}h^2\right)(\lambda_2 - b)\right\}; \\ m_1 &= \frac{q}{24J}\left[3\lambda_2^2 - 2b\lambda_2 - b^2 + 12\left(L^2 - \frac{2}{5}h^2\right)\right]; \\ m_2 &= \frac{q}{6J}(\lambda_2 - b); \\ m_3 &= \frac{q}{3J}. \end{aligned}\right\} \qquad (V.36)$$

3) Considering the bend of an arm with a crack under the action of concentrated force Q (see Fig. 63) we find

$$\begin{gathered} \sigma_y^+(t_0) = -\frac{Q}{2J}(2L-d)(2t_0 + a - b + \\ + \lambda_1 + \lambda_2)\sqrt{\frac{(t_0-\lambda_1)(t_0-\lambda_2)}{(t_0+a)(t_0-b)}}. \end{gathered} \qquad (V.37)$$

Here $\lambda_1 \leq t_0 \leq \lambda_2$, where $\lambda_2 = b$ (for this problem compression stresses act in region $x > 0$), and λ_1 is found from equation (V.32) for $\lambda_2 = b$,

$$\lambda_1 = \frac{1}{3}a. \qquad (V.37a)$$

Calculation of complex potentials $\Phi(z)$ and $\Omega(z)$. Into formulas (V.12) and (V.13) it is necessary to place values of $p(t_0)$ in

accordance with (V.29)-(V.37), and then to calculate corresponding Cauchy integrals. As a result of such calculations we establish the following formulas for functions $\Phi(z)$ and $\Omega(z)$.

1. Considering the pure bend of a strip with a crack (see Fig. 61)

$$\Phi(z) = \frac{M}{6J}(3z - 2b)\sqrt{\frac{z-\lambda_2}{z-b}} - \frac{M}{4J}z;$$
$$\Omega(z) = \Phi(z) + \frac{M}{2J}z \quad \left(\lambda_2 = -\frac{1}{3}b\right). \tag{V.38}$$

2. Considering the bend of a strip of evenly distributed load of intensity q (see Fig. 62)

$$\Phi(z) = \frac{1}{2}(m_3 z^3 + m_2 z^2 + m_1 z + m_0)\sqrt{\frac{z-\lambda_2}{z-b}} - \frac{q}{8J}\left[z^3 + \left(L^2 - \frac{7}{5}h^2\right)z + \frac{2}{3}h^3\right]; \tag{V.39}$$
$$\Omega(z) = \Phi(z) + \frac{q}{4J}\left[z^3 + \left(L^2 - \frac{7}{5}h^2\right)z + \frac{2}{3}h^3\right], \tag{V.39a}$$

where λ_2 is determined from equation (V.35), and coefficients m_j ($j = 0, 1, 2, \ldots$) are represented by formulas (V.36).

$$\Phi(z) = -\frac{1-i\rho_0}{2}\cdot\frac{Q}{J}(2L-d)\left(z + \frac{2}{3}a\right)\sqrt{\frac{z-\lambda_1}{z+a}} + \frac{1}{\sqrt{(z+a)(z-b)}}\left\{c_3 z^3 + c_2 z^2 + c_1 z + c_0 + \frac{(1-i\rho_0)Q}{16J}(2L-d) \times [8z^3 - 4(b-a)z - (b+a)^2]\right\} - \frac{iQ}{8J}[3z^2 + 2i(2L-d)z - 2h^2]; \tag{V.40}$$
$$\Omega(z) = \Phi(z) + \frac{iQ}{4J}[3z^2 + 2i(2L-d)z - 2h^2], \tag{V.40a}$$

where $\lambda_1 = 1/3\ a$; coefficients c_j ($j = 0, 1, 2, 3$) are determined by formulas (V.18).

On the basis of formulas (V.4), (V.5) and (V.38)-(V.40) we can find the components σ_x, σ_y, τ_{xy} of the stress tensor in the

neighborhood of the examined cracks considering curvature of the strip. Thus, in the plane of location of the crack, i.e., when $y = 0$, we have the following.

1. Considering pure bend of a strip with a crack (see Fig. 61)

$$\left.\begin{aligned}
\sigma_y(x, 0) &= B\left(\frac{x}{b} - \frac{2}{3}\right)\sqrt{\frac{x-\lambda_2}{x-b}} \qquad \left(B = \frac{Mb}{J}\right); \\
\sigma_x(x, 0) &= \sigma_y(x, 0) - \frac{M}{J}x \qquad \left(J = \frac{4\tau h^3}{3}\right); \\
\tau_{xy}(x, 0) &= 0,
\end{aligned}\right. \qquad (V.41)$$

where $\lambda_2 = -1/3\, b$; $-h \leq x \leq -1/3\, b$ or $b \leq x \leq h$.

2. Considering bend of a strip with a crack by an evenly distributed load (see Fig. 62)

$$\left.\begin{aligned}
&\sigma_y(x, 0) = (m_3x^3 + m_2x^2 + m_1x + m_0)\sqrt{\frac{x-\lambda_2}{x-b}}; \\
&\sigma_x(x, 0) = \sigma_y(x, 0) - \frac{q}{2J}\left[x^3 + \left(L - \frac{7}{5}h^2\right)x + \frac{2}{3}h^2\right]; \\
&\tau_{xy}(x, 0) = 0,
\end{aligned}\right| \qquad (V.42)$$

where $-h \leq x \leq \lambda_2$, $b \leq x \leq h$, λ_2 is determined from equation (V.36), and coefficients m_0, m_1, m_2, and m_3 are expressed by equalities (V.36).

3. Considering the bend of an arm with a crack (see Fig. 63)

$$\left.\begin{aligned}
&\sigma_y(x, 0) = -\frac{Q}{3J}(2L - d)(3x + 2a)\sqrt{\frac{x-\lambda_2}{x+a}}; \\
&\sigma_x(x, 0) = \sigma_y(x, 0) + \frac{Q}{J}(2L - d)x; \\
&\tau_{xy}(x, 0) = \rho_0\sigma_y(x, 0) - \frac{Q}{2J}\frac{1}{\sqrt{(x+a)(x-b)}}\Big\{x^3 - \\
&-\frac{1}{2}(b-a)x^2 - \left[\frac{(b+a)^2}{8} + h^2\right]x - \frac{(b-a)}{16}[(b + \\
&+a)^2 - 8h^2] + \rho_0\Big[-2(2L-d)x^2 + (2L-d)(b - \\
&-a)x + \frac{1}{4}(2L-d)(b+a)^2\Big]\Big\},
\end{aligned}\right| \qquad (V.43)$$

where $-h \le x \le -a$, $\lambda_1 \le x \le h$, $\lambda_1 = 1/3\ a$.

In order to obtain a certain idea about the damping rate of the field of elastic stresses caused by the presence of the crack in the bent strip, on Fig. 64 in accordance with formulas (V.41) graphs (solid lines) are constructed for the change of components of stress tensor $\sigma_x\ (x, 0)$, $\sigma_y\ (x, 0)$, $\tau_{xy}\ (x, 0)$ depending upon distance x when $x \ge b$ and $x \le -1/3\ b$. The dotted line shows the change of stresses

$$\sigma_y^0(x, 0) = B\frac{x}{b}, \quad \sigma_x^0(x, 0) = 0, \quad \tau_{xy}^0(x, 0) = 0 \qquad \left(B = \frac{Mb}{J}\right), \qquad (V.44)$$

which appear during pure bending of the same strip, but without a crack.

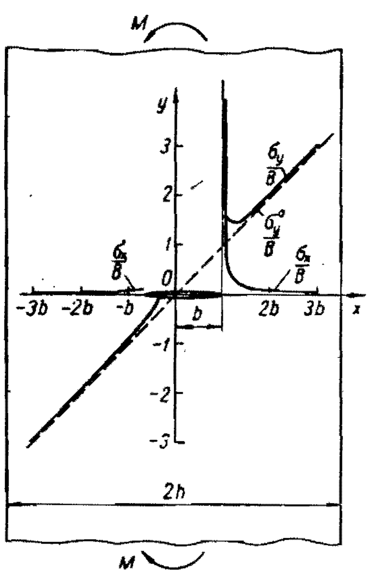


Fig. 64.

Analyzing the graphs shown in Fig. 64, it is easy to note that when $x > 2b$ the perturbed state of strain can be assumed as coinciding with the undisturbed, i.e., with the state of strain which is determined by functions (V.7).

Study of the stressed-deformed and maximum state of strips (balls) weakened by randomly oriented slotted cracks and curvilinear

holes is also of interest. Similar problems are examined in works [23, 83, 84, 88, 106, 149]. Determination of the stressed-deformed state in a strip weakened by a peaked hole in the form of a hypocycloid can be done on the basis of results of this section and Section 2 Chapter IV (see also work [106]).

3. Calculation of Limit Loads During the Bend of a Strip with Cracks

For determination of the limit values of an external load during the bend of a strip with cracks, we will use equations given in Chapter III:

$$\lim_{r\to 0}\{\sqrt{r}\,\sigma^{*}_{\beta}(r,\beta_{*})\} = \frac{K}{\pi}; \tag{V.45}$$

$$\lim_{r\to 0}\left\{\sqrt{r}\left(\frac{\partial\sigma_{\beta}}{\partial\beta}\right)_{\beta=\beta_{*}}\right\} = 0, \tag{V.45a}$$

where r, β — polar coordinates with origin at tip of crack; σ_{β} (r, β) — component of stress tensor in this system of coordinates (angle β is read counterclockwise from the plane of location of the crack).

Obviously (see Chapter III), if the plane of the crack is the plane of symmetry of the external loads, then crack propagation should go along this plane, and, consequently, angle $\beta_{*} = 0$. In such a case

$$\lim_{r\to 0}\{\sqrt{r}\,\sigma^{*}_{y}(r,0)\} = \frac{K}{\pi}, \tag{V.46}$$

where $r = x - x_j$, xj — abscissa of the examined tip of the crack in rectangular system of coordinates xOy, when the crack is located along axis Ox (see Fig. 61).

<u>Strip with centrally located crack</u>. Let us find the limit load considering bend of a strip with centrally located crack (Figs. 61 and 62). To solve such a problem elastic stresses in the neighborhood of the crack can be determined on the basis of formulas (V.4), (V.5) and (V.38), (V.39) (see also formulas (V.41) and (V.42).

Thus, stresses σ_y (x, 0) are expressed by the following formulas:

1) for pure bend of a strip with a crack (see Fig. 61)

$$\sigma_y(x, 0) = \frac{Mb}{J}\left(\frac{x}{b} - \frac{2}{3}\right)\sqrt{\frac{x-\lambda_2}{x-b}} \qquad \left(\lambda_2 = -\frac{1}{3}b\right), \tag{V.47}$$

where M – bending moment; J – moment of inertia of strip; $b \leq x \leq h$;

2) for the bend of a strip with a crack (see Fig. 62) under the action of evenly distributed pressure of intensity q

$$\sigma_y(x, 0) = (m_3x^3 + m_2x^2 + m_1x + m_0)\sqrt{\frac{x-\lambda_2}{x-b}} \quad (b \leqslant x \leqslant h), \tag{V.48}$$

where values λ_2 and m_0, m_1, m_2, m_3 are determined accordingly by equalities (V.35) and (V.36).

Thus, limit loads for such problems can be determined on the basis of formulas (V.47) and (V.48) according to equation (V.46). Substituting for each of the examined examples the values of elastic stresses σ_y (x, 0) into equation (V.45a), and then carrying out in this equation passage to the limit when $x \to x_j = b$, or which is the same, when $r \to 0$, we obtain formulas for determination of the limit load ($M = M_*$, $q = q_*$) during the bend of a strip with a crack:

$$M_* = \frac{3\sqrt{3}J}{2b} \cdot \frac{K}{\pi\sqrt{b}} \qquad \left(J = \frac{4\tau h^3}{3}\right); \tag{V.49}$$

$$q_* = \frac{48JK(\sqrt{b-\lambda_2})^{-1}}{\pi(b+\lambda_2)\left[5\lambda_2^2 - 2b\lambda_2 + 5b^2 + 12\left(L^2 - \frac{2}{5}h^2\right)\right]}, \tag{V.50}$$

where parameter λ_2 is determined from the equation

$$35\lambda_2^4 - 20b\lambda_2^3 - 6b^2\lambda_2^2 - 4b^3\lambda_2 - 5b^4 + $$
$$+ 24\left(L^2 - \frac{2}{5}h^2\right)(3\lambda_2^2 - 2b\lambda_2 - b^2) = 0.$$

From formulas (V.49) and (V.50) we have directly the following: for our problems (Figs. 61, 62) the limit values of loads $M = M_*$ and $q = q_*$ do not depend on the parameter a, and other things being equal depend only on parameter b, i.e., on the characteristic linear dimension of that part of the crack which is located in the region of tensile stresses.

Strip with crack located in the region of tensile stresses. Let us assume that a strip with a noncentrally located macroscopic crack is subjected to bending by moments M (Fig. 65) or evenly distributed pressure q (Fig. 66) and let us assume that in each of these cases the crack is located wholly in the zone of tensile stresses. Let us determine the limit load, i.e., find the value M_* and q_* respectively.

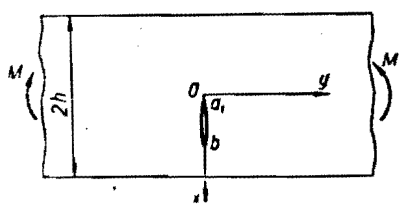


Fig. 65.

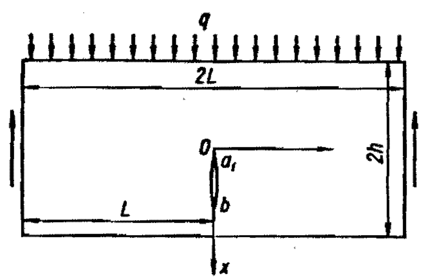


Fig. 66.

In examining such a problem we will use rectangular system of cartesian coordinates xOy (Figs. 65 and 66), assuming that the crack is located along axis Ox when $a_1 \leq x \leq b$, where a_1, b – the abscissas of the ends of the crack ($b > a_1 > 0$, length of the crack $2l_1 = b - a_1$).

By the condition of the problem the crack is located in the region of tensile stresses, therefore when the strip is being bent, the opposite sides of the crack do not touch each other, and consequently, there are no contact stresses between them. Having this in mind view and using results of Section 2 of this chapter, for our problems it is easy to calculate complex potentials $\Phi(z)$ and $\Omega(z)$. These functions will be obtained on the basis of formulas (V.7)-(V.9) and (V.12)-(V.17), if we set $p(t) = 0$ and $a = -a_1$:

1) considering pure bend of a strip with a noncentrally located crack (see Fig. 65)

$$\begin{gathered}\Phi(z)=\frac{M}{16J}\times\\ \times\frac{8z^2-4(a_1+b)z-(b-a_1)^2}{\sqrt{(z-a_1)(z-b)}}-\\ -\frac{M}{4J}z;\\ \Omega(z)=\Phi(z)+\frac{M}{2J}z,\end{gathered} \tag{V.51}$$

where

$$J=\frac{4\tau h^3}{3};$$

2) considering the bend of a strip with noncentrally located crack under the action of evenly distributed load q (see Fig. 66)

$$\begin{gathered}\Phi(z)=f(z)-\frac{q}{8J}\left[z^3+\left(L^2-\frac{7}{5}h^2\right)z+\frac{2}{3}h^3\right];\\ \Omega(z)=f(z)+\frac{q}{8J}\left[z^3+\left(L^2-\frac{7}{5}h^2\right)z+\frac{2}{3}h^3\right],\end{gathered} \tag{V.52}$$

where

$$\begin{gathered}f(z)=\frac{q}{96J\sqrt{(z-a_1)(z-b)}}\Big\{16z^4-8(a_1+b)z^3+\\ +2\left[12\left(L^2-\frac{2}{5}h^2\right)-(b-a_1)^2\right]z^2-(a_1+b)\left[12\left(L^2-\frac{2}{5}h^2\right)+\right.\\ \left.+(b-a_1)^2\right]z-\frac{(b-a_1)^2}{8}\left[24\left(L^2-\frac{2}{5}h^2\right)+\right.\\ \left.+5a_1^2+6a_1b+5b^2\right]\Big\}.\end{gathered} \tag{V.52a}$$

On the basis of equations (V.4) and (V.5) and also formulas (V.51) and (V.52) we find:

1) for the first example (see Fig. 65)

$$\sigma_y(x, 0) = \frac{M}{8J} \cdot \frac{8x^2 - 4(a_1 + b)x - (b - a_1)^2}{\sqrt{(x - a_1)(x - b)}};$$
$$\sigma_x(x, 0) = \sigma_y(x, 0) - \frac{M}{J}x, \quad \tau_{xy}(x, 0) = 0, \tag{V.53}$$

where $-h \leq x \leq a_1$, $b \leq x \leq h$;

2) for the second example (see Fig. 66)

$$\sigma_y(x, 0) = 2f(x), \quad \tau_{xy}(x, 0) = 0;$$
$$\sigma_x(x, 0) = \sigma_y(x, 0) - \frac{q}{2J}\left[x^3 + \left(L^2 - \frac{7}{5}h^2\right)x + \frac{2}{3}h^3\right], \tag{V.54}$$

where $-h \leq x \leq a_1$; $b \leq x \leq h$; $f(x)$ is determined by formula (V.52a) when $z = x + i0$.

Just as is done in examining maximum equilibrium of strips weakened by centrally located cracks, on the basis of formulas (V.46), (V.53) and (V.54) we have

$$M_* = \frac{8J}{3b + a_1} \cdot \frac{K}{\pi\sqrt{b - a_1}}; \tag{V.55}$$

$$q_* = \frac{384JK\left(\sqrt{b - a_1}\right)^{-1}}{\pi\left\{35b^3 + 15a_1b^2 + 9a_1^2b + 5a_1^3 + 24(3b + a_1)\left(L^2 - \frac{2}{5}h^2\right)\right\}}. \tag{V.56}$$

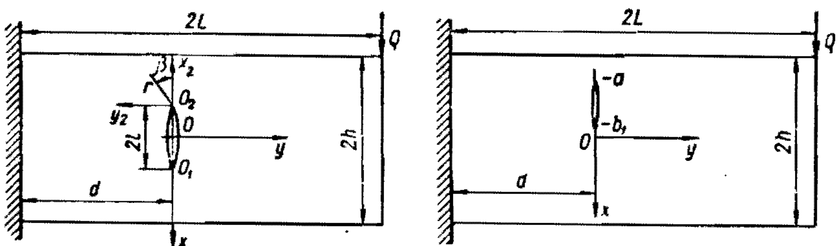
From formulas (V.49), (V.50), (V.55) and (V.56) it follows that with an increase of parameter b, i.e., increase of crack length ($2l = b + a$, $2l_1 = b - a_1$) values of the limit loads M_*, q_* decrease and consequently, there occurs unstable propagation of cracks. Thus, in our problems the limit load $M = M_*$, $q = q_*$, after which the initial crack becomes a moving equilibrium crack, coincides with the limit load. Analogous problems about maximum equilibrium of a strip weakened by two collinear cracks of unequal length located in the zone of tensile stresses are examined in work [126].

4. Continuation. Determination of Limit Load During the Bending of an Overhang Beam with a Crack

During the bending of an overhang beam with macroscopic crack of length $2l = b + a$ (Fig. 63) the plane of location of the crack is not the plane of symmetry of the acting loads. Tangential stresses in this plane are not equal to zero (see formula (V.43)), and consequently, this plane is not generally the plane of initial propagation of the crack when external loads reach their limit level.

Thus, in the case of the transverse bending of an overhang beam with cracks, the determination of initial direction of crack propagation and calculate the limit values of external load must be done on the basis of equations (V.45). For this, as it is known, in the beginning it is necessary to calculate the components of the elastic stress tensor in the neighborhood of that end of the crack which is located in the zone of higher tensile stresses.

Considering what was said, we will determine the values of limit load for certain cases of our problem.



a) b)

Fig. 67.

The case of a centrally located crack (Fig. 67a). Let us consider the diagram on Fig. 67a. In this case on the basis of formulas (V.43) when $a = b = l$, we have the following complex potentials:

$$\Phi(z) = -\frac{Q(2L-d)}{6J}(1 - i\rho_0)(3z + 2l)\sqrt{\frac{z-\lambda_1}{z+l}} + \frac{iQ}{4J}\cdot\frac{1}{\sqrt{z^2-l^2}}\left[z^3 - 2\rho_0(2L-d)z^2 - \left(\frac{l^2}{2}+h^2\right)z + \rho_0 l^2(2L-d)\right] - \frac{iQ}{8J}\left[3z^2 + 2i(2L-d)z - 2h^2\right];$$
$$\Omega(z) = \Phi(z) + \frac{iQ}{4J}\left[3z^2 + 2i(2L-d)z - 2h^2\right], \qquad (V.57)$$

where $\lambda_1 = 1/3\, l$; $J = 4\tau h^3/3$ (remaining designations are shown on Fig. 67a).

For our problem the biggest tensile stresses will be in the neighborhood of the end of the crack O_2. In a small neighborhood of of point O_2 (Fig. 67a) components (σ_r, σ_β, $\tau_{r\beta}$) of the elastic stress tensor in polar system of coordinates (r, β) can be determined on the basis of formulas (III.96), (III.99), (III.103) and functions (V.57). In particular we have

$$\sigma_\beta(r,\beta) = \frac{1}{4\sqrt{2r}}\left\{k_{1,j}\left[3\cos\frac{\beta}{2} + \cos\frac{3}{2}\beta\right] - \right.$$
$$\left. - 3k_{2,j}\left[\sin\frac{\beta}{2} + \sin\frac{3}{2}\beta\right] + 0(1). \right. \tag{V.58}$$

Here coefficients $k_{1,j}$ and $k_{2,j}$ are determined from equation (see Chapter IV)

$$k_{1,j}\sqrt{\frac{2}{r}}\cos\frac{\beta}{2} - k_{2,j}\sqrt{\frac{2}{r}}\sin\frac{\beta}{2} + 0(1) = 4\,\mathrm{Re}\,\Phi_j(z_j), \tag{V.59}$$

where $\Phi_j(z_j) = \Phi(z_{0j} + z_j e^{i\omega j})$; z_{0j} – affix of the tip of the crack; $z_j = re^{i\beta}$ – complex variable in new system of coordinates ($r\beta$); ω_j – angle of rotation of new system of coordinates with respect to the old.

For tip O_2 (Fig. 67a) we have

$$z_{0j} = z_{02} = -l; \quad \omega_j = \omega_2 = \pi; \quad z_2 = re^{i\beta}.$$

Using these equalities and formulas (V.57) we will find

$$\Phi_2(z_2) = \Phi(-l - re^{i\beta}) = \frac{Q\sqrt{l}}{4J}\sqrt{\frac{2}{r}}\,e^{-i\frac{\beta}{2}}\left\{\frac{4(1-i\rho_0)(2L-d)\,l}{3\sqrt{6}} + \right.$$
$$\left. + \frac{l}{2}\left[h^2 - \frac{l^2}{2} - \rho_0 l\,(2L-d)\right]\right\} + 0(1). \tag{V.60}$$

Putting the expression $\Phi_2(z_2)$ in the right side of equality (V.59) and comparing coefficients $k_{1,2}$ and $k_{2,2}$ at identical harmonics in the left and rigit side of the equality, we obtain

$$k_{1,2} = \frac{4Ql\sqrt{l}\,(2L-d)}{3\sqrt{6}\,J},$$

$$k_{2,2} = -\frac{4Q\sqrt{l}}{J}\left[-\frac{\rho_0 l\,(2L-d)}{3\sqrt{6}} + \frac{2h^2 - 2\rho_0 l\,(2L-d) - l^2}{16}\right]. \quad \text{(V.61)}$$

Thus, on the basis of formulas (V.58) and (V.61) one can determine the principal part of the stresses σ_β (r, β) in the neighborhood of the examined tip O_2 through centrally located crack in a beam subjected to transverse bending (see Fig. 67a).

Using these formulas, and also equations (V.45) and (V.46) for determination of the limit load in the case of transverse bend of an overhang beam we will obtain the following relationships:

$$k^*_{1,2}\left(3\cos\frac{\beta_*}{2} + \cos\frac{3}{2}\beta_*\right) - 3k^*_{2,2}\left(\sin\frac{\beta_*}{2} + \sin\frac{3}{2}\beta_*\right) = \\ = \frac{4\sqrt{2}K}{\pi}; \quad \text{(V.62)}$$

$$k_{1,2}\left(\sin\frac{\beta_*}{2} + \sin\frac{3}{2}\beta_*\right) + k_{2,2}\left(\cos\frac{\beta_*}{2} + 3\cos\frac{3}{2}\beta_*\right) = 0, \quad \text{(V.63)}$$

where coefficients $k_{1,2}$ and $k_{2,2}$ are determined by formulas (V.60), and $k^*_{1,2}$ and $k^*_{2,2}$ by the same formulas, setting $Q = Q_*$.

Solving equation (V.63), we find [see (III.107)] the formula for calculation of angle β_*, determining the direction of initial crack propagation in our case:

$$\beta_* = 2\operatorname{arctg}\frac{1 \pm \sqrt{1 + 8n^2}}{n}, \quad \text{(V.64)}$$

where the "+" corresponds to values of $k_{1,2} < 0$, and the "-" to values of $k_{1,2} > 0$;

$$n = \frac{k_{2,2}}{k_{1,2}} = \rho_0 - \frac{3\sqrt{6}}{16}\cdot\frac{2h^2 - 2\rho_0(2L-d)\,l^2}{l\,(2L-d)}.$$

Placing in equation (V.62) values of coefficients $k^*_{1,2}$ and $k^*_{2,2}$ in accordance with equalities (V.61) when $Q = Q_*$ and carrying out

simple transformations, we obtain a formula for determination of limit load $Q = Q_*$:

$$Q_* = \frac{3\sqrt{3}KJ}{2\pi\sqrt{l}\,(2L-d)\,l\cos^2\frac{\beta_*}{2}\left\{\cos\frac{\beta_*}{2} - 3\sin\frac{\beta_*}{2}\left[-\rho_0 + \frac{3\sqrt{6}}{16}\cdot\frac{2h^2 - 2l\rho_0(2L-d) - l^2}{(2L-d)\,l}\right]\right\}}, \tag{V.65}$$

where angle β_* is determined by formula (V.64).

In every concrete case of the bend of an overhang beam with macroscopic crack (see Fig. 67a) for given values of parameters L, d, l, and ρ_0 by formulas (V.65) it is possible to calculate the value of limit load $Q = Q_*$ after which the crack starts to spread along the cross section of the beam.

If we consider that during the transverse bending of an overhang beam with macroscopic centrally located crack the direction of initial propagation of the crack practically coincides with the plane of location of the crack, i.e., consider that in the first approximation $\beta_* \approx 0$, then from equality (V.65) it is easy to obtain the following simple approximate formula:

$$Q_* \approx \frac{3\sqrt{3}J}{2\sqrt{2}(2L-d)\,l}\cdot\frac{\sqrt{2}K}{\pi\sqrt{l}}, \tag{V.66}$$

where

$$J = \frac{4\tau h^3}{3}.$$

Formula (V.66) is obtained by other means in work [86].

Let us compare the values of limit loads $Q = Q_*$, calculated by formulas (V.65) and (V.66), for the following relationships between parameters L, d, h, l, and ρ_0:

$$d = L;\quad h = ml\,(m = 2, 3, 4, 5, \ldots);\quad \rho_0 = 0;\quad \rho_0 = 0{,}01;\quad \rho_0 = 0{,}1;$$
$$L = 10^n l\ \ (n = 1, 2, 3, \ldots).$$

In this concrete case formulas (V.65) and (V.66) can be transformed respectively to the form

$$Q_* = \frac{KJ}{\pi l^2 \sqrt{l}} \Lambda_1 (m, n, \beta_*); \tag{V.66a}$$

$$Q_* \approx \frac{KJ}{\pi l^2 \sqrt{l}} \Lambda_2 (n), \tag{V.66b}$$

where

$$\Lambda_1 (m, n, \beta_*) = \frac{3\sqrt{3}}{2 \cdot 10^n \cos^2 \frac{\beta_*}{2} \left\{ \cos \frac{\beta_*}{2} - 3 \sin \frac{\beta_*}{2} \left[-\rho_0 + \frac{3\sqrt{6}}{16} \cdot \frac{2m^2 - 2\rho_0 10^n - 1}{10^n} \right] \right\}}; \tag{V.66c}$$

$$\Lambda_2 (n) = \frac{3\sqrt{3}}{2 \cdot 10^n} \approx 2{,}598 \cdot 10^{-n}.$$

Value of function $\Lambda_1(m, n, \beta_*)$, and also angle β_* for different relationships between parameters L, d, h, l, and ρ_0 are given in Table 16.

Table 16.

ρ_0	$h = 2l$			$h = 5l$	
	n	β_*	Λ_1	β_*	Λ_1
0	1	30° 32′	0,2233	62° 19′	0,07951
	2	3° 40′	0,02528	23° 16′	0,02366
	3	0° 22′	0,002532	2° 34′	0,002531
0,01	1	29° 16′	0,2231	62° 13′	0,07950
	2	1° 29′	0,02530	2° 32′	0,02366
	3	—1° 50′	0,002532	0° 23′	0,002532
0,1	1	14° 19′	0,2306	61° 35′	0,07802
	2	—17° 19′	0,02574	0° 23′	0,02525
	3	—20° 03′	0,002636	—18° 15′	0,003288

Comparing values of $\Lambda_1(m, n, \beta_*)$ and $\Lambda_2(n)$, it is not difficult to note that when L >> l formula (V.66) can be used to evaluate the value limit load in the case of transverse bending of an overhang beam with centrally located crack (Fig. 67a).

The case when the crack is located in a zone of tensile stresses. Let us assume that an isolated macrocrack is wholly in a zone of

tensile stresses, when the overhang beam is subjected to transverse bending by force Q (Fig. 67b). As a result of arguments analogous to those which are given above, and on the basis of formulas (V.10), (V.12)-(V.14) and (V.18), in which one should set $p(t) \equiv 0$, $b = -b_1$ ($b_1 > 0$), we will obtain the following expressions of $\Phi(z)$ and $\Omega(z)$ for our problem:

$$\Phi(z) = \frac{iQ}{64J\sqrt{(z+a)(z+b_1)}}\{16z^3 + [8(a+b_1) + 32i(2L-d)]z^2 + \\ + [16i(a+b_1)(2L-d) - 16h^2 - 2(a-b_1)^2]z + (a+b_1)[(a-b_1)^2 - \\ - 8h^2] - 4i(2L-d)(a-b_1)^2\} - \frac{iQ}{8J}\left[3z^2 + 2i(2L-d)z - 2h^2\right], \tag{V.67}$$
$$\Omega(z) = \Phi(z) + \frac{iQ}{8J}[3z^2 + 2i(2L-d)z - 2h^2].$$

On the basis of formulas (V.59) and (V.67) we will calculate coefficients $k_{1,2}$, $k_{2,2}$ intensity of elastic stresses in the neighborhood of the tip of the crack with abscissa $x = -a$, i.e., when $z = -a - re^{i\beta}$, where $r \ll a$. Formulas for determination of these coefficients have the form

$$k_{1,2} = \frac{Q\sqrt{a-b_1}}{4\sqrt{2}J}(2L-d)(3a+b_1),$$
$$k_{2,2} = \frac{Q\sqrt{a-b_1}}{16\sqrt{2}J}(8h^2 - 5a^2 - 2ab_1 - b_1^2). \tag{V.68}$$

Hence, according to equations (V.62) and (V.63) we find

$$Q_* = \frac{8JK}{\pi\sqrt{a-b_1}\cos^2\frac{\beta_*}{2}}\left\{(2L-d)(3a+b_1)\cos\frac{\beta_*}{2} - \right.$$
$$\left. -3\left(2h^2 - \frac{5a^2+2ab_1+b_1^2}{4}\right)\sin\frac{\beta_*}{2}\right\}^{-1}, \tag{V.69}$$

where angle β_* can be determined by formula (V.64) under the condition that coefficients $k_{1,2}$ and $k_{2,2}$ are calculated by formulas (V.68).

Assuming for our problem that initial crack propagation occurs along the plane of its location, i.e., assuming in the first approximation $\beta_* \approx 0$, from formula (V.69) we have [124]

$$Q_* \approx \frac{8J}{(3a+b_1)(2L-d)} \cdot \frac{K}{\pi\sqrt{a-b_1}}. \tag{V.70}$$

In conclusion let us note that problems about determination of limit load for a strip (beam) with a crack located in the zone of tensile stresses during the bending of the strip (see Figs. 65, 66, 67b) have been investigated also on the basis of a δ_k - model in work [87]. In the special case when the length of the crack is macroscopic (rather developed) from results of work [87] follow formulas (V.55), (V.56), (V.70).

Footnotes

[1]For this class of problems it is possible to say beforehand that the region of contact between the sides of the crack will always start with the extreme (end) points of the crack which is in the region of compression stresses. Consequently, either parameter λ_1 or λ_2 will be known beforehand.

[2]Subsequently values of the coefficients of polynomial $R_n(z)$ are not needed.

[3]The method of solving integral equations of form (V.30) is expounded in works [69, 104].

C H A P T E R VI

CERTAIN RELATIONSHIPS OF THE STATICS OF A THREE-DIMENSIONAL ELASTIC BODY

1. Formulation of Problems of the Theory of Elasticity in Shifts

Three-dimensional problems of the theory of elasticity and methods of solving them have been formulated in a number of textbooks and monographs on the theory of elasticity (see for example, [33, 67, 89, 90, 140, 157, 160]). Below only certain relationships are derived for the statics of a three-dimensional elastic body, necessary later in the study of maximum equilibrium of brittle three-dimensional bodies weakened by internal or surface disk-like cracks.

Initial equations. Let us assume that an elastic body occupies certain volume V_s and on its surface S external stresses are assigned. Solving the problem of the theory of elasticity for such a body, i.e., determination of stresses and deformations in the body, reduces to finding three functions:

$$u = u(x, y, z); \quad v = v(x, y, z) \text{ and } w = w(x, y, z), \tag{VI.1}$$

which are the components (projections) of the vector of elastic shifts of points of a deformable body accordingly along axis Ox, Oy, and Oz in rectangular system of cartesian coordinates Oxyz, connected with the examined body.

In the absence of mass forces functions u, v and w must satisfy boundary conditions assigned on the surface of the body and the following equilibrium equations (Lamé equations):

$$\left.\begin{aligned}\Delta u+\frac{1}{1-2\nu}\cdot\frac{\partial\Theta}{\partial x}=0;\\ \Delta v+\frac{1}{1-2\nu}\cdot\frac{\partial\Theta}{\partial y}=0;\\ \Delta w+\frac{1}{1-2\nu}\cdot\frac{\partial\Theta}{\partial z}=0,\end{aligned}\right\} \tag{VI.2}$$

$$\left(\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2};\quad \Theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

where ν – Poisson's ratio; Δ – three-dimensional Laplacian operator; Θ – relative cubic expansion.

If on all surface S, bounding examined body V, shifts are assigned, then boundary conditions for functions u, v, and w have the following form:

$$u=u_s,\quad v=v_s,\quad w=w_s. \tag{VI.3}$$

When on all the surface of a body external stresses ($F_n^{(s)}$) are assigned then boundary conditions for functions u, v, and w are expressed so:

$$\left.\begin{aligned}&G\left[2\left(\frac{\partial u}{\partial x}+\frac{\nu}{1-2\nu}\Theta\right)\cos(n,x)+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\cos(n,y)+\right.\\&\left.+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\cos(n,z)\right]=F_{nx}^{(s)};\\&G\left[\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\cos(n,x)+2\left(\frac{\partial v}{\partial y}+\frac{\nu}{1-2\nu}\Theta\right)\cos(n,y)+\right.\\&\left.+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\cos(n,z)\right]=F_{ny}^{(s)};\\&G\left[\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\cos(n,x)+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\cos(n,y)+\right.\\&\left.+2\left(\frac{\partial w}{\partial z}+\frac{\nu}{1-2\nu}\Theta\right)\cos(n,z)\right]=F_{nz}^{(s)}.\end{aligned}\right\} \tag{VI.4}$$

Here n — external normal to surface S; $F_{nx}^{(s)}$, $F_{ny}^{(s)}$, and $F_{nz}^{(s)}$ — projections of external stresses onto coordinate axes; G — shear modulus, $G = E/2(1 + \nu)$.

When on one part of the surface of the body (S_1) shifts are assigned, and on the other ($S_2 = S - S_1$) external stresses (so-called mixed problem of the theory of elasticity), then on surface S_1 it is necessary to satisfy boundary conditions (VI.3), and on surface S_2 boundary conditions (VI.4).

If functions u, v, and w are determined for a deformable body, then components σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx} of the stress tensor in any point of the body can be calculated on the basis of Hooke's law and Cauchy relationships, i.e., by the formulas

$$\begin{aligned} \sigma_x &= 2G\left(\frac{\partial u}{\partial x} + \frac{\nu}{1-2\nu}\Theta\right), & \tau_{xy} &= G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); \\ \sigma_y &= 2G\left(\frac{\partial v}{\partial y} + \frac{\nu}{1-2\nu}\Theta\right), & \tau_{yz} &= G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right); \\ \sigma_z &= 2G\left(\frac{\partial w}{\partial z} + \frac{\nu}{1-2\nu}\Theta\right), & \tau_{zx} &= G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right). \end{aligned} \tag{VI.5}$$

Consequently, the problem of the theory of elasticity in shifts reduces to solving system of differential equations (VI.2), satisfying assigned boundary conditions (VI.3) and (VI.4) on the surface of the deformable body. However, to find a solution to system of equations (VI.2), satisfying assigned boundary conditions on the surface of a body of arbitrary configuration is difficult. This can be done only for certain of the simplest forms of a body and the simplest forms of external influences (stresses or shifts) on its boundary.

Subsequently during the study of maximum equilibrium of brittle bodies with internal disk-like cracks it is necessary to solve problems of the theory of elasticity for a three-dimensional space with flat surfaces; below some of these problems are examined.

Elastic space with flat boundary (crack). Let us consider elastic body V_s, occupying all of a three-dimensional space with the exception of region S plane z = 0 (region S is examined as a cut in an elastic body). Let us assume that to the boundary of the region is applied normal pressure of intensity q(x, y), and in points at infinity of the body there are no stresses. It is required to determine stressed σ_z (x, y, 0) in plane z = 0 outside region S and components w (x, y, 0) of the vector of displacements for points of region S.

For our problem the external load is symmetric with respect to the plane z = 0, therefore, obviously, in this plane there are no tangential stresses, and the shifts in the direction of axis Oz are equal to zero. For all points of the plane z = 0 outside region S. Consequently, we have the following conditions:

$$\tau_{zx} = \tau_{zy} = 0 \text{ when } z = 0; \tag{VI.6}$$

$$\sigma_z(x, y, 0) = -q(x, y) \text{ in region S}; \tag{VI.7}$$

$$w(x, y, 0) = 0 \text{ in region } S_\infty. \tag{VI.8}$$

where S_∞ is exterior of region S of the plane z = 0.

Differential equilibrium equations (VI.2) and conditions of equality to zero of tangential stresses in the plane z = 0, i.e., conditions (VI.6) will be satisfied if components of the vector of elastic shifts are represented in the following form:

$$u = \varphi_1 - \frac{z}{2(1-\nu)} \cdot \frac{\partial \varphi_3}{\partial x}; \quad v = \varphi_2 - \frac{z}{2(1-\nu)} \cdot \frac{\partial \varphi_3}{\partial y}; \quad w = \varphi_3 - \frac{z}{2(1-\nu)} \cdot \frac{\partial \varphi_3}{\partial z}, \tag{VI.9}$$

where ϕ_1, ϕ_2 and ϕ_3 – harmonic functions of viarables x, y, z, interconnected by the equalities

$$\frac{\partial \varphi_1}{\partial z} = -\frac{1-2\nu}{2(1-\nu)} \cdot \frac{\partial \varphi_3}{\partial x}; \quad \frac{\partial \varphi_2}{\partial z} = -\frac{1-2\nu}{2(1-\nu)} \cdot \frac{\partial \varphi_3}{\partial y}. \tag{VI.10}$$

Consequently, for a given value of function w in the plane $z = 0$ the formulated problem reduces to determination of harmonic function ϕ_3 by its value on the boundary of half-space $z \geq 0$, i.e., to resolution of Dirichlet's problem.

According to the above arguments and on the basis of formulas (VI.5), (VI.9) and (VI.10) components σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx} of the stress tensor can be represented as:

$$\frac{2(1-\nu^2)}{E}\sigma_z = \frac{\partial\varphi_3}{\partial z} - z\frac{\partial^2\varphi_3}{\partial z^2}; \tag{VI.11}$$

$$\left.\begin{aligned}
&\frac{2(1-\nu^2)}{E}\sigma_y = 2\nu\frac{\partial\varphi_3}{\partial z} + 2(1-\nu)\frac{\partial\varphi_2}{\partial y} - z\frac{\partial^2\varphi_3}{\partial y^2},\\
&\frac{2(1-\nu^2)}{E}\sigma_x = 2\nu\frac{\partial\varphi_3}{\partial z} + 2(1-\nu)\frac{\partial\varphi_1}{\partial x} - z\frac{\partial^2\varphi_3}{\partial x^2},\\
&\frac{2(1-\nu^2)}{E}\tau_{zx} = -z\frac{\partial^2\varphi_3}{\partial x\partial z}; \quad \frac{2(1-\nu^2)}{E}\tau_{zy} = -z\frac{\partial^2\varphi_3}{\partial y\partial z},\\
&\frac{2(1-\nu^2)}{E}\tau_{xy} = (1-\nu)\left(\frac{\partial\varphi_1}{\partial y} + \frac{\partial\varphi_2}{\partial x}\right) - z\frac{\partial^2\varphi_3}{\partial x\partial y},
\end{aligned}\right\} \tag{VI.12}$$

where, according to equalities (VI.10), we have

$$\varphi_1 = \frac{(1-2\nu)}{2(1-\nu)}\int_z^\infty \frac{\partial\varphi_3}{\partial x}dz; \quad \varphi_2 = \frac{(1-2\nu)}{2(1-\nu)}\int_z^\infty \frac{\partial\varphi_3}{\partial y}dz \tag{VI.13}$$

when $z \geq 0$.

Thus, if to body V_S with cut S in the plane $z = 0$ external loads are applied, symmetric with respect to the plane $z = 0$, then solving the problem of the theory of elasticity reduces to finding one harmonic function ϕ_3 in assigned conditions in the plane $z = 0$.

From formulas (VI.9), (VI.11) and (VI.12) it is easy to conclude that in the plane $z = 0$ conditions (VI.6) are satisfied unconditionally and component σ_z of the stress tensor and component w of the displacement vector are expressed through harmonic function ϕ_3 by the following equalities:

$$\begin{aligned}
&\varphi'_{3z}(x, y, 0) = \frac{2(1-\nu^2)}{E}\cdot\sigma_z(x, y, 0),\\
&\varphi_3(x, y, 0) = w(x, y, 0),
\end{aligned} \tag{VI.14}$$

where

$$\varphi'_{3x}(x, y, 0) = \left[\frac{\partial\varphi_3(x, y, z)}{\partial x}\right]_{z=+0}.$$

Hence, according to equalities (VI.7) and (VI.8), for determination of function ϕ_3 (x, y, z) harmonic in region V_S we have these mixed boundary conditions:
in region S

$$\varphi'_{3z}(x, y, 0) = -\frac{2(1-\nu^2)}{E} q(x, y); \tag{VI.15}$$

in region S_∞

$$\varphi_3(x, y, 0) = 0.$$

Our problem of the theory of elasticity reduced to a mixed problem of the theory of the potential of determination of function ϕ_3 (x, y, z), harmonic in the half-space $z \geq 0$, vanishing in points at infinity of the body and satisfying in the plane z = 0 mixed boundary conditions (VI.15). Consequently, for determination of function ϕ_3 (x, y, z) requires solving the mixed problem of the theory of a potential for the half-space $z \geq 0$. To do this we present the desired function in the form of the integral

$$\varphi_3(x, y, z) = \frac{1-\nu^2}{\pi E} \lim_{S\to\infty} \iint_{(S)} \frac{q(\xi, \eta)\, d\xi d\eta}{r}, \tag{VI.16}$$

where

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + z^2.$$

This integral, as it is known, constitutes a harmonic function in the whole space except region S of the plane z = 0. In the plane z = 0 the following equality holds:

$$\left.\frac{\partial\varphi_3}{\partial z}\right|_{z=+0} = \begin{cases} -\frac{2(1-\nu^2)}{E} q(x, y) & \text{in region S;} \\ 0 & \text{outside region S,} \end{cases} \tag{VI.17}$$

where ϕ'_{3z} (x, y, z) is a harmonic function.

On the basis of formulas (VI.16) and (VI.17) we conclude that harmonic function ϕ_3 (x, y, z) can be determined by the assigned boundary value in the plane z = 0 thus:

$$\varphi_3(x, y, z) = -\frac{1}{2\pi} \lim_{S\to\infty} \frac{\partial}{\partial z} \iint_{(S)} \frac{\varphi_3(\xi, \eta, +0)}{r}\, d\xi d\eta. \tag{VI.18}$$

Differentiating formula (VI.18) with respect to z, we find

$$\frac{\partial \varphi_3}{\partial z} = -\frac{1}{2\pi} \lim_{S\to\infty} \frac{\partial^2}{\partial z^2} \iint_{(S)} \frac{\varphi_3(\xi, \eta, +0)}{r}\, d\xi d\eta. \tag{VI.19}$$

Taking into account that the integral in the first part of equality (VI.19) is a harmonic function and considering relationship (VI.14), we obtain [64]

$$\sigma_z(x, y, 0) = \frac{E}{4\pi(1-\nu^2)} \lim_{S\to\infty} \Delta_{xy} \iint_{(S)} \frac{w(\xi, \eta, 0)\, d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}, \tag{VI.20}$$

where Δ_{xy} is a two-dimensional Laplacian operator,

$$\Delta_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Consequently, for the examined body V_s in the plane z = 0 function w (x, y, 0) is known, then the stress component σ_z (x, y, 0) is determined by the formula (VI.20). If, however, in the plane z = 0 stresses σ_z (x, y, 0) = -q (x, y) are known, then the shift component of w (x, y, 0) is determined by formulas (VI.14) and (VI.16).

When in plane z = 0 mixed boundary conditions are assigned, as conditions (VI.7)-(VI.8) are, then, using formula (VI.20), we obtain the following equation for determination of function w (x, y, 0) in region S (outside region S function w (x, y, 0) = 0):

$$q(x, y) = \frac{-E}{4\pi(1-\nu^2)} \Delta_{xy} \iint_{(S)} \frac{w(\xi, \eta, 0)\, d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}, \tag{VI.21}$$

where points x, y belong to region S of the plane z = 0.

Solving equation (VI.21) for assigned region S and load q (x, y) and satisfying conditions of the continuity of function w (x, y, 0) on the contour of region S, we will find finally the solution of the examined problem.

2. Contact Problems for a Half-Space and Determination of Stresses Near a Plane Crack

Let us assume that an elastic body occupies all the space on one side of a certain plane, for example, the plane z = 0. Such a body we will call an elastic half-space, and the plane z = 0 the surface or boundary of the half-space. Axis Oz of rectangular system of cartesian coordinates Oxyz is directed inside the half-space so that for all points of the body $z \geq 0$. Furthermore, we will designate by S and S_∞ respectively the internal and external region of the plane z = 0, into which the surface of the elastic half-space is separated by a certain closed contour L (Fig. 68).

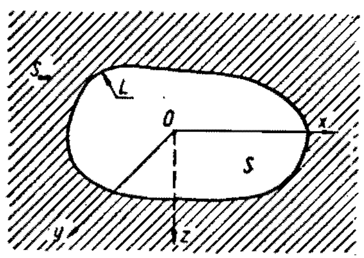


Fig. 68.

Let us assume that on the surface of the elastic half-space in region S_∞ is located an absolutely rigid body – a stamp with a flat base (this body can be considered linked with an elastic support, but so that in the plane z = 0 slipping of points of the elastic body is possible, and in region S act stresses $\sigma_z (x, y, 0) = -q (x, y)$. It is assumed that there are no frictional forces between the touching bodies.

It is necessary to determine stresses σ_z (x, y, 0) in region S_∞, i.e., under the base of the stamp, and shifts w (x, y, 0) for points of the plane z = 0 outside the stamp (in region S).

To solve these problems on the surface of the elastic half-space we have the following boundary conditions:
when z = 0

$$\begin{gathered}\tau_{zx} = \tau_{zy} = 0; \\ \sigma_z(x, y, 0) = -q(x, y) \text{ in region } S; \\ w(x, y, 0) = 0 \text{ in region } S_\infty\end{gathered} \qquad \text{(VI.22)}$$

(in points at infinity of the body there are no stresses and shifts).

Boundary conditions (VI.22) of the formulated contact problem for an elastic half-space wholly coincide with boundary conditions (VI.6)-(VI.8) for the problem about distribution of stresses in a three-dimensional elastic body with a plane cracked cut, when forces applied to the body are symmetric with respect to the plane of location of the crack and in the region of the crack lead to normal pressure q (x, y). Hence we conclude that these problems are equivalent.

Consequently, determination of stresses in a three-dimensional infinite body with internal plane crack S in points of the region S_∞, (when the body is loaded by a system of forces symmetric with respect to the plane of locations of the crack) is equivalent to the solution of the contact problem for a half-space, when it is required to determine pressures (stresses) under the base of a flat ideally smooth stamp located on the surface of an elastic half-space in regions S_∞, appearing as a result of pressure q (x, y) in region S.

If the crack is located in region S_∞ and it is necessary to determine stresses in region S of a three-dimensional body, then using analogous arguments we conclude that the problem reduces to determination of pressure under the base of a flat ideally smooth stamp, which is located on the surface of an elastic half-space in

region S, and external forces are applied in region S_∞ (Fig. 68). However, we must consider that for such a problem the external load q (x, y) should approach zero as $x^2 + y^2 \to \infty$, inasmuch as in formulating the problem it was assumed that in points at infinity of the body there are no stresses (shifts).

Reasoning the same way, it is easy to establish that shifts w (x, y, 0) of the sides of the crack S in a three-dimensional body correspond to shifts w (x, y, 0) of the surface of the elastic half-space located outside the region of the stamp under the action of an assigned load q (x, y) in this region.

Mixed (contact) problems of the theory of elasticity for a half-space were investigated by many authors (see work [33]). The results can be used in calculating stresses in three-dimensional bodies with cracks. Below we will examine the problem for a half-space when region S is a circle.

Problem for a circular region. It is known [33, 65] that if on the surface of an elastic half-space a stamp with flat ideally smooth base is freely located, having in its plan (region S of plane z = 0) the form of a circle, and outside the region of contact, i.e., in the region S_∞ of the plane z = 0, there acts normal pressure q (x, y), then normal stresses σ_z (x, y, 0) appearing in this case under the base of the stamp are expressed by the formula

$$\sigma_z(x, y, 0) = \frac{1}{\pi^2 \sqrt{a^2 - x^2 - y^2}} \iint\limits_{(S_\infty)} \frac{q(\xi, \eta)\sqrt{\xi^2 + \eta^2 - a^2}}{(x-\xi)^2 + (y-\eta)^2} d\xi d\eta, \qquad \text{(VI.23)}$$

where

$$x^2 + y^2 \leqslant a^2; \quad \xi^2 + \eta^2 > a^2.$$

Noticing further that in formula (VI.23) variables x, y, ξ, η enter symmetrically, we conclude that if an ideally smooth stamp with a flat base is located on the surface of an elastic half-space outside the circle $x^2 + y^2 = a^2$, and normal pressures q (x, y) act in the region $x^2 + y^2 < a^2$, then stresses σ_z (x, y, 0) when $x^2 + y^2 \geq a^2$ are determined from the equality

$$\sigma_z(x, y, 0) = \frac{1}{\pi^2\sqrt{x^2+y^2-a^2}}\iint\limits_{(S)} \frac{q(\xi, \eta)\sqrt{a^2-\xi^2-\eta^2}}{(x-\xi)^2+(y-\eta)^2}\, d\xi d\eta, \qquad \text{(VI.24)}$$

where

$$x^2+y^2 \geqslant a^2; \quad \xi^2+\eta^2 < a^2.$$

Thus, rupture stresses σ_z (x, y, 0), appearing in a three-dimensional body with a plane (round in plan) crack, when the sides of the crack are under normal pressure q (x, y), are determined by formula (VI.23) or (VI.24). These formulas can be simplified by introducing polar system of coordinates (r, β, ρ, α) and assuming that

$$\begin{aligned} &\xi = \rho\cos\alpha, \quad \eta = \rho\sin\alpha, \quad \xi^2+\eta^2 = \rho^2; \\ &x = r\cos\beta, \quad y = r\sin\beta, \quad x^2+y^2 = r^2, \end{aligned} \qquad \text{(VI.25)}$$

where

$$0 \leqslant \alpha \leqslant 2\pi; \quad 0 \leqslant \beta \leqslant 2\pi.$$

In this case formula (VI.24) can be written as:

$$\sigma_z(r, \beta, 0) = \frac{1}{\pi^2\sqrt{r^2-a^2}}\int_0^a\int_0^{2\pi} \frac{\sqrt{a^2-\rho^2}\,q(\rho, \alpha)\,\rho\,d\alpha\,d\rho}{r^2+\rho^2-2r\rho\cos(\alpha-\beta)} \qquad (r \geqslant a). \qquad \text{(VI.26)}$$

Formula (VI.23) can be written the same way.

Let us assume that pressure is expressed by the formula

$$q[\rho, \alpha] = f_{1m}(\rho)\sin m\alpha + f_{2m}(\rho)\cos m\alpha, \qquad \text{(VI.27)}$$

where m is positive integer.

Placing expression (VI.27) in (VI.26), we obtain

$$\begin{aligned} \sigma_z(r, \beta, 0) = \frac{1}{\pi^2\sqrt{r^2-a^2}}\int_0^a \rho\sqrt{a^2-\rho^2}\,[f_{1m}(\rho)\,I_{1m}(\rho, \beta) + \\ + f_{2m}(\rho)\,I_{2m}(\rho, \beta)]\,d\rho, \\ I_{1m}(\rho, \beta) = \int_0^{2\pi} \frac{\sin m\alpha\,d\alpha}{r^2+\rho^2-2r\rho\cos(\alpha-\beta)}, \end{aligned} \qquad \text{(VI.28)}$$

where

$$I_{2m}(\rho, \beta) = \int_0^{2\pi} \frac{\cos m\alpha d\alpha}{r^2 + \rho^2 - 2r\rho\cos(\alpha - \beta)}. \tag{VI.28a}$$

Integrals (VI.28) are easy to calculate on the basis of the theory of residues [156]. These integrals can be represented in the following form:

$$I_{1m}(\rho, \beta) = \frac{2\pi}{r^2 + \rho^2} \cdot \frac{(1 - \sqrt{1 - \mu^2})^m}{\mu^m \sqrt{1 - \mu^2}} \sin m\beta = I_m^{(0)}(\rho) \sin m\beta;$$

$$I_{2m}(\rho, \beta) = \frac{2\pi}{r^2 + \rho^2} \cdot \frac{(1 - \sqrt{1 - \mu^2})^m}{\mu^m \sqrt{1 - \mu^2}} \cos m\beta = I_m^{(0)}(\rho) \cos m\beta, \tag{VI.29}$$

where

$$\mu = \frac{2r\rho}{r^2 + \rho^2}.$$

Placing expressions (VI.29) in formula (VI.28), we find

$$\sigma_z(r, \beta, 0) = \frac{1}{\pi^2 \sqrt{r^2 - a^2}} \{A_m(r) \sin m\beta + B_m(r) \cos m\beta\}, \tag{VI.30}$$

where

$$A_m(r) = \int_0^a \sqrt{a^2 - \rho^2} I_m^{(0)}(\rho) f_{1m}(\rho) \rho d\rho,$$

$$B_m(r) = \int_0^a \sqrt{a^3 - \rho^2} I_m^{(3)}(\rho) f_{2m}(\rho) \rho d\rho.$$

If normal pressure $q[\rho, \alpha]$ is axisymmetrically applied to the sides of the crack, i.e., in expansion (VI.27) we have $m = 0$, then for such a case on the basis of formula (VI.30) we obtain

$$\sigma_z(r, \beta, 0) = \frac{2}{\pi \sqrt{r^2 - a^2}} \int_0^a \frac{\rho \sqrt{a^2 - \rho^2}}{r^2 - \rho^2} f_0(\rho) d\rho, \tag{VI.31}$$

where

$$r \geqslant a.$$

Formula (VI.23) can be converted in the same way.

3. Determination of Normal Shifts of the Sides of a Disk-Shaped Crack

A shift w (x, y, z) of the sides of a round (disk-shaped) crack in a three-dimensional body, when inside crack there acts normal pressure q (x, y) can be determined by equation (VI.21). However, in the general case of a load q (x, y) the solution of this equation meets great difficulties. These difficulties sometimes can be overcome if the solution of one axisymmetrical contact problem of the theory of elasticity is applied to an elastic half-space.

Let us solve such a problem for the case when into an elastic half-space $z \geq 0$ is imbedded a stamp in the form of an arbitrary solid of revolution whose axis of symmetry is normal to the surface of the half-space and coincides with axis Oz. Moreover, we assume that there are no frictional forces between the touching bodies. The stamp touches the elastic half-space along a certain circle of radius a.

Inside this circle the "set" of the stamp is known, i.e., shift w (x, y, 0), which due to the axial symmetry of the stamp will be a function of only polar radius r:

$$w(x, y, 0) = w(r, 0) \quad (r = \sqrt{x^2 + y^2}). \tag{VI.32}$$

Outside the contact region the surface of the elastic half-space is free from external stresses and, consequently, stresses σ_z (r, β, 0) = 0 when r > a.

Under the base of the stamp appears normal pressure p (x, y), which is also a function of only distance r, i.e.,

$$p(x, y) = p(r) \text{ when } r \leqslant a, \tag{VI.33}$$

The problem consists in determination of pressure p(r). On the basis of formulas (VI.14) and (VI.16), and also according to conditions (VI.32) for determination of the unknown pressure we have the following integral equation:

$$f(r) = \iint_{(S_a)} \frac{p(\rho)\, ds}{r_{MN}}, \tag{VI.34}$$

where S_a – region of circle $x^2 + y^2 \leqslant a^2$; $f(r) = \frac{\pi E}{1-\nu^2} w(r, 0)$;

$$r_{MN} = \sqrt{(x-\xi)^2 + (y-\eta)^2} \; -$$

distance between points M and N of region S_a; ds – element of area.

Let us introduce [63] new variables y, α, since this is shown on Fig. 69. Then

$$\rho = \sqrt{r^2 \sin^2 \alpha + y^2}; \quad ds = r_{MN} dy d\alpha. \tag{VI.35}$$

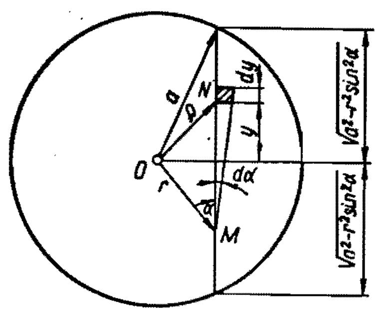


Fig. 69.

In new variables (VI.35) equation (VI.34) can be written as:

$$f(r) = 4 \int_0^{\pi/2} \int_0^{\sqrt{a^2 - r^2 \sin^2 \alpha}} p\left(\sqrt{r^2 \sin^2 \alpha + y^2}\right) dy d\alpha. \tag{VI.36}$$

In work [63] it is shown that the general solution of integral equation (VI.36), continuous in region $0 \leq r \leq a$ and finite on its contour, is expressed by the formula

$$p(r) = -\frac{1}{\pi^2} \int_0^{\pi/2} \int_0^{\sqrt{a^2 - r^2} \sin \alpha} \Delta_r f\left(\sqrt{a^2 \sin^2 \alpha + y^2}\right) dy d\alpha, \tag{VI.37}$$

where $\Delta_r f$ — two-dimensional Laplacian operator,

$$\Delta_r f = \frac{1}{r} f' + f''.$$

Let us examine now the problem posed at the beginning of the section. Let us assume that on the surface z = 0 of an elastic half-space in the region $S_\infty (x^2 + y^2 \geq a^2)$ vertical shifts w (x, y, 0) are equal to zero, and in the region $S_a (x^2 + y^2 < a^2)$ there acts normal axisymmetrical pressure q(r). It is necessary to determine vertical shifts w (x, y, 0) of points of the surface z = 0 in region S_a.

For this problem formula (VI.20) can be written as:

$$\Delta_r \iint\limits_{(S_a)} \frac{f(\rho)\, ds}{r_{MN}} = -4\pi^2 q(r) - \Delta_r \iint\limits_{(S_\infty)} \frac{f(\rho)\, ds}{r_{MN}}, \qquad \text{(VI.38)}$$

where designations are the same as in formula (VI.34).

In region S_∞ by the conditions of the problem $f(\rho) = 0$. Consequently, the integral in the right side of equation (VI.38) is equal to zero, and the integral in the left side of this equation can be transformed as is done with formula (VI.34). As a result of such transformations equation (VI.38) takes the form

$$\Delta_r \int\limits_0^{\pi/2} \int\limits_0^{\sqrt{a^2 - r^2 \sin^2\alpha}} f\left(\sqrt{r^2 \sin^2\alpha + y^2}\right) dy\, d\alpha = -\pi^2 q(r) \quad (r \leqslant a). \qquad \text{(VI.39)}$$

Furthermore, the following identity is known [63]

$$\Delta_{xy} \iint\limits_{(S_a)} \frac{f(\rho)\, ds}{r_{MN}} = \iint\limits_{(S_a)} \frac{\Delta_{xy} f(\rho)\, ds}{r_{MN}}. \qquad \text{(VI.40)}$$

Using identity (VI.40) in equation (VI.39) and taking into account formulas (VI.36) and (VI.37), and then again (VI.40), we obtain the following expression for determination of shifts:

$$w(r, 0) = \frac{4(1 - \nu^2)}{\pi E} \int\limits_0^{\pi/2} \int\limits_0^{\sqrt{a^2 - r^2 \sin\alpha}} q\left(\sqrt{r^2 \sin^2\alpha + y^2}\right) dy\, d\alpha. \qquad \text{(VI.41)}$$

In the region $r \leq a$ these shifts are continuous, and when $r = a$ they turn into zero.

This formula was first time set up by M. Ya. Leonov in work [63] in the solution of an axisymmetrical contact problem of the theory of elasticity for a half-space.

CHAPTER VII

LIMITING EQUILIBRIUM OF A BRITTLE BODY WITH A PLANE FRACTURE CIRCULAR IN PLANE

1. Uniaxial Extension of a Three-Dimensional Body with an Internal Round Fracture

Let us assume that in a uniform brittle body is an internal fracture circular in plan. The crack can be represented in the form of a flattened spheroid whose axis coincides with axis Oz, and the plane of crack with plane xOy(z = 0) of rectangular system of cartesian coordinates Oxyz (Fig. 70). Let us assume that in points at infinity of this body are applied monotonically increasing tensile stresses of intensity p ($\sigma_z(x, y, \infty) = p$), directed perpendicularly to the plane of the crack. It is necessary to determine limit stresses $p = p_*$.

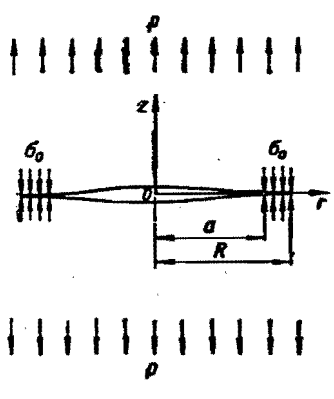


Fig. 70.

Let us designate by a the radius of the initial crack (Fig. 70). If we assume that this radius is large enough for the crack to be

considered macroscopic (see Section 6, Chapter I), the problem could be solved by the power method of Griffith, as is done in a work of Saha [216]. However, we will not make such assumptions, but will try to find a solution to these problems for an arbitrary radius a of a disk-shaped crack. Solving the generalized problem, we will be able to indicate a boundary of applicability of the Saha solution and also to clarify certain peculiarities of the rupture of brittle bodies with minute cracks.[1]

Our problem about the rupture of a brittle body with an internal round crack of radius a will be solved within the bounds formed by a reference model (δ_k-model) of a brittle body (see Chapter I). In accordance with the properties of this model in the neighborhood of the contour of a real crack in a field of tensile stresses a region of weakened interpartial connections always will be formed, which are examined as ultramicroscopic cracks. Opposite sides of such cracks are attracted to each other with constant stress σ_0, if the distance between them does not exceed a certain value δ_k, and do not interact if this distance exceeds δ_k. The size of δ_k is connected with stresses σ_0 and effective surface energy γ of a given material by the equality:

$$\delta_k = \frac{2\gamma}{\sigma_0}. \qquad \text{(VI.1)}$$

For our problem (Fig. 70) the region of the ultramicroscopic crack will be ring-shaped

$$a \leqslant r \leqslant R \qquad \text{(VII.2)}$$

around the contour of the initial r = a, where r is the polar radius of points of the deformable body located in the region of the crack; R is the radius of the boundary between the elastically deformed material and the microscopic crack.

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[1]Such a generalized setting of the problem is proposed a work of M. Ya. Leonov and the author [71], where the basic results expounded in this section are found.

Tensile stresses established by methods of linear theory of elasticity in a deformable body with microscropic cracks must not exceed the limit of brittle strength σ_0, i.e., they must be finite.

Further, in accordance with properties of the reference model of a brittle body, external stresses $p = p_*$ will be maximum, i.e, with such stresses the crack goes into a moving equilibrium state if the following equality holds:

$$2w(a, 0, p_*) = \delta_k, \tag{VII.3}$$

where $w(r, 0, p)$ are normal shifts of the sides of a crack ($r \leq R$) during the action of preassigned stresses p.

Thus, the problem about the rupture of a brittle body with an internal (round in plan) crack (Fig. 70) leads to determination of the function $w(r, 0, p)$ in the region of the crack ($r \leq R$) and parameter R under corresponding boundary conditions, i.e., to the subsequent problem of the theory of elasticity.

Let us assume that in an elastic three-dimensional body is a plane (round in plan) crack of radius R. On the surface of this crack act normal stresses

$$\sigma_z(r, 0) = \begin{cases} 0 & \text{when } r \leqslant a; \\ \sigma_0 & \text{when } a < r \leqslant R, \end{cases} \tag{VII.4}$$

and in points of infinity of the body act stresses $\sigma_z(r, \infty) = p$. It is required to determine the shift of the sides of the crack and stresses outside cracks which satisfy the condition of being finite.

<u>Determination of shifts and stresses in the plane of the crack</u>. If we subtract from the state of strain appearing in a body with a crack when $r \leq R$ (see Fig. 70) and under boundary conditions (VII.4) the uniform state of strain $\sigma_z = p$ appearing in a body without a crack when the body is subjected to uniaxial extension by stresses of

intensity p, then as a result we will obtain a certain auxiliary state of strain in a body with a crack, which vanishes on infinity, and on the surface of the crack is characterized by the following conditions:

$$q(r)=\begin{cases} p & \text{when } r \leqslant a; \\ p-\sigma_0 & \text{when } a<r\leqslant R, \end{cases} \qquad \text{(VII.5)}$$

where $q(r) = -\sigma_z(r, 0)$ is the normal axisymmetrical pressure on the surface of the crack for the auxiliary problem.

During uniaxial extension of an unbounded body without a crack, there are no shifts of points of the plane $z = 0$ (plane of symmetry), therefore the shifts of the sides of the crack for the initial problem coincide with the shifts of the sides of the crack for the auxiliary problem, and components of the stress tensor for the initial problem differ from components of the stress tensor for the auxiliary problem by a constant $\sigma_z = p$.

Consequently, shifts w(r, 0) of the sides of the crack for our problem (Fig. 70) can be determined by formula (VI.41), i.e., by the formula

$$\frac{\pi E}{4(1-\nu^2)}w(r,0)=\int_0^{\pi/2}\int_0^{\sqrt{R^2-r^2\sin^2\alpha}} q\left(\sqrt{r^2\sin^2\alpha+y^2}\right)dy\,d\alpha, \qquad \text{(VII.6)}$$

where function q is the normal pressure on the surface of the crack.

For convenicence of calculation formula (VII.6) should be transformed, considering (see Fig. 70) that

$$\rho^2=r^2\sin^2\alpha+y^2.$$

After transformations we obtain

$$\frac{\pi E}{4(1-\nu^2)}w(r,0)=\int_0^{\pi/2}\int_{r\sin\alpha}^{R\sin\alpha}\frac{q(\rho)\,\rho\,d\rho}{\sqrt{\rho^2-r^2\sin^2\alpha}}\,d\alpha, \qquad \text{(VII.7)}$$

where $q(\rho)$ is normal pressure on the surface of a round crack ($0 \leq \rho \leq R$) represented by the equality (VII.5).

According to relationships (VII.5), the internal integral in formula (VII.7) takes the following values:

1) if $0 \leq r \leq a$, then

$$\int_{r\sin\alpha}^{R\sin\alpha} \frac{q(\rho)\rho d\rho}{\sqrt{\rho^2 - r^2\sin^2\alpha}} = \begin{cases} p\sqrt{R^2 - r^2}\sin\alpha & \text{when } 0 \leqslant \alpha \leqslant \arcsin\frac{a}{R}; \\ (p - \sigma_0)\sqrt{R^2 - r^2}\sin\alpha + \sigma_0\sqrt{a^2 - r^2\sin^2\alpha} & \\ & \text{when } \arcsin\frac{a}{R} \leqslant \alpha \leqslant \frac{\pi}{2}; \end{cases}$$

2) if $a \leq r \leq R$, then

$$\int_{\sin\alpha}^{R\sin\alpha} \frac{q(\rho)\rho d\rho}{\sqrt{\rho^2 - r^2\sin^2\alpha}} = \begin{cases} p\sqrt{R^2 - r^2}\sin\alpha & \text{when } 0 \leqslant \alpha \leqslant \arcsin\frac{a}{R}; \\ (p - \sigma_0)\sqrt{R^2 - r^2}\sin\alpha + \sigma_0\sqrt{a^2 - r^2\sin^2\alpha} & \\ & \text{when } \arcsin\frac{a}{R} \leqslant \alpha \leqslant \arcsin\frac{a}{r}; \\ (p - \sigma_0)\sqrt{R^2 - r^2}\sin\alpha & \\ & \text{when } \arcsin\frac{a}{r} \leqslant \alpha \leqslant \frac{\pi}{2}. \end{cases}$$

Using these expressions and formula (VII.7), we find when $0 \leq r \leq a$

$$\frac{\pi E}{4(1-\nu^2)} w(r, 0) = \sqrt{R^2 - r^2}\left(p - \sigma_0\frac{\sqrt{R^2 - a^2}}{R}\right) + \\ + \sigma_0 \int_{\arcsin\frac{a}{R}}^{\pi/2} \sqrt{a^2 - r^2\sin^2\alpha}\, d\alpha; \tag{VII.8}$$

when $a \leq r \leq R$

$$\frac{\pi E}{4(1-\nu^2)} w(r, 0) = \sqrt{R^2 - r^2}\left(p - \sigma_0\frac{\sqrt{R^2 - a^2}}{R}\right) + \\ + \sigma_0 \int_{\arcsin\frac{a}{R}}^{\arcsin\frac{a}{r}} \sqrt{a^2 - r^2\sin^2\alpha}\, d\alpha. \tag{VII.8a}$$

Differentiating formula (VII.8a) with respect to r, we obtain

$$\frac{\pi E}{4(1-\nu^2)}\cdot\frac{dw(r,0)}{dr}=\frac{-r}{\sqrt{R^2-r^2}}\left(p-\frac{\sigma_0}{R}\sqrt{R^2-a^2}\right)-$$
$$-r\sigma_0\int\limits_{\arcsin\frac{a}{R}}^{\arcsin\frac{a}{r}}\frac{\sin^2\alpha d\alpha}{\sqrt{a^2-r^2\sin^2\alpha}}. \qquad \text{(VII.9)}$$

For final determination of function w(r, 0) it is necessary still to find the value of parameter R in formulas (VII.8) and (VII.8a). The calculate stresses $\sigma_z(r, 0)$ when $r \geq R$. These stresses in accordance with what we have said and on the basis of formula (VI.31) can be represented as:

$$\sigma_z(r,0)\frac{2}{\pi\sqrt{r^2-R^2}}\int\limits_0^R\frac{\rho\sqrt{R^2-\rho^2}}{r^2-\rho^2}q(\rho)\,d\rho+p, \qquad \text{(VII.10)}$$

where $r \geq R$; $q(\rho)$ – normal axisymmetrical pressure, represented by equality (VII.5).

Then, using formulas (VII.5) and (VII.10), we find

$$\sigma_z(r,0)=\frac{2}{\pi\sqrt{r^2-R^2}}\left\{p\left(R-\sqrt{r^2-R^2}\operatorname{arctg}\frac{R}{\sqrt{r^2-R^2}}\right)-\right.$$
$$\left.-\sigma_0\left(\sqrt{R^2-a^2}-\sqrt{r^2-R^2}\operatorname{arctg}\frac{\sqrt{R^2-a^2}}{\sqrt{r^2-R^2}}\right)\right\}+p, \qquad \text{(VII.11)}$$

where $r \geq R$.

According to formula (VII.11), for an arbitrary value of parameter R stresses $\sigma_z(r, 0)$ tend to infinity if $r \to R$. But, in accordance with properties of the δ_k- model of a brittle body, parameter R should ensure that stresses in the elastic body outside the defect (crack) are finite. On the basis of formula (VII.11) we find that the necessary condition for this is conversion into zero of the expression in the braces of the formula when r = R. Then to determine parameter R we have the equality

$$\lim_{r\to R+0}\left\{p\left(R-\sqrt{r^2-R^2}\operatorname{arctg}\frac{R}{\sqrt{r^2-R^2}}\right)-\right.$$
$$\left.-\sigma_0\left(\sqrt{R^2-a^2}-\sqrt{r^2-R^2}\operatorname{arctg}\frac{\sqrt{R^2-a^2}}{\sqrt{r^2-R^2}}\right)\right\}=0.$$

Hence we find

$$pR-\sigma_0\sqrt{R^2-a^2}=0. \qquad \text{(VII.12)}$$

By condition (VII.12) the value of parameter R is determined. Moreover, form formula (VII.11) we find that

$$\lim_{r\to R+0}\sigma_z(r,0)=\sigma_0, \qquad \text{(VII.13)}$$

i.e., in this case stresses on the contour of the crack not only are finite, but also are continuous.

On the basis of equality (VII.12) formula (VII.18) and (VII.8a) can be given the following form:

$$w(r,0)=\frac{4(1-\nu^2)}{\pi E}\sigma_0\int_{\arcsin\frac{a}{R}}^{\psi}\sqrt{a^2-r^2\sin^2\alpha}\,d\alpha \qquad (r\leqslant R), \qquad \text{(VII.14)}$$

where when $0 \leq r \leq a \psi = \frac{\pi}{2}$, when $a \leq r \leq R \psi = \arcsin\frac{a}{r}$.

When condition (VII.12) holds, according to formula (VII.9) we have

$$\lim_{r\to R}\left(\frac{dw(r,0)}{dr}\right)=0 \quad \left(\lim_{r\to R}w(r,0)\neq 0\right), \qquad \text{(VII.15)}$$

i.e., in this case (as also in the case of the two-dimensional problem) we have a smooth closing of the opposite sides of the crack when r = R. It is simple also to note that on the basis of formula (VII.9) condition (VII.12) can be examined as the requirement for smooth closing of the sides of a crack on its contour when r = R.

Determination of limit stresses. On the basis of equality (VII.12) to determine the boundary ($r = R$) between the elastic region of the body and the crack we have the formula

$$R = \frac{a}{\sqrt{1 - \left(\frac{p}{\sigma_0}\right)^2}}. \qquad \text{(VII.16)}$$

This formula expresses the dependence of parameter R on external stresses p when $p \leq p_*$.

Further, using formula (VII.14), it is easy to find

$$2w(a, 0) = \frac{8(1 - \nu^2)}{\pi E} a\sigma_0 \left(1 - \frac{a}{R}\right). \qquad \text{(VII.17)}$$

Thus, on the basis of relationships (VII.3), (VII.16) and (VII.17) to determine limit load $p = p_*$ in the case of our problem (see Fig. 70) we have the following equation:

$$\sqrt{1 - \left(\frac{p_*}{\sigma_0}\right)^2} = 1 - \frac{a_*}{a}, \qquad \text{(VII.18)}$$

where

$$a_* = \frac{\pi E \delta_k}{8(1 - \nu^2)\sigma_0}. \qquad \text{(VII.19)}$$

It is obvious that equation (VII.18) has a real solution for unknown function p_* if the radius of the crack $a \geq a_*$. In such a case from this equation we find

$$p_* = \sigma_0 \sqrt{\frac{2a_*}{a}} \sqrt{1 - \frac{a_*}{2a}} \qquad (a \geqslant a_*). \qquad \text{(VII.20)}$$

Differentiating formula (VII.20) with respect to a, we obtain

$$\frac{dp_*}{da} < 0 \text{ when } a > a_*.$$

Hence, and also on the basis of formula (VII.20) we conclude that values of limit stresses p_* decrease with an increase of the radius of the crack a when $a > a_*$, i.e., in this case after external stresses p reach $p = p_*$ unstable crack propagation begins in the body. Consequently, limit load p_*, determined by formula (VII.20), is the breaking load for a brittle body with a disk-shaped crack of radius $a \geq a_*$.

Let us consider the case when $a < a_*$. Let us note that according to formula (VII.16) when external stresses $p = \sigma_0$ act on a body, boundary R of the region of weakened interpartial bonds goes to infinity. In this case (when $a < a_*$, $p = \sigma_0$, $R = \infty$) from formula (VII.17) we find

$$2w(a, 0) = 8\frac{1-\nu^2}{\pi E}\sigma_0 a < 8\frac{1-\nu^2}{\pi E}\sigma_0 a_* = \delta_k. \qquad \text{(VII.21)}$$

Relationship (VII.21) shows that if a brittle body has an internal disk-shaped crack of radius $a < a_*$, then during the extension of such a body by external stresses p, equal even to the limit brittle strength σ_0 of the material, on the contour of such a crack a rupture front does not appear. Consequently, the size of the limit stresses does not depend on the presence in the body of another crack if $a < a_*$. A brittle body with such a defect possesses the strength of faultless material, i.e.,

$$p_* = \sigma_0 \text{ when } a < a_*. \qquad \text{(VII.22)}$$

Thus, on the basis of relationships (VII.20) and (VII.22) to determine the size of the rupture stresses p during uniaxial estension of a brittle body with an internal crack we obtain the following formula:

$$p_* = \begin{cases} \sigma_0 & \text{when} \quad a < a_*; \\ \sigma_0\sqrt{\frac{2a_*}{a}}\sqrt{1-\frac{a_*}{2a}} & \text{when} \quad a \geqslant a_*. \end{cases} \qquad \text{(VII.23)}$$

where

$$a_* = \frac{\pi E \delta_k}{8(1-\nu^2)\sigma_0}.$$

We will assume that the examined round crack is rather large (macroscopic) and for such a crack the following relationships are valid:

$$\frac{a_*}{a} \ll 1; \qquad \sqrt{1-\frac{a_*}{2a}} \approx 1.$$

In this case from formula (VII.23) we get the Saha formula [216]:

$$p_* = \sigma_0 \sqrt{\frac{2a_*}{a}} = \sqrt{\frac{\pi E \gamma}{2(1-\nu^2)a}}. \qquad \text{(VII.24)}$$

Note: In work [99] a solution is given to the Saha problem for a nonuniform brittle body constituting two joined half-spaces with different elastic properties, when in the bonding plane is a round macrocrack. In solving the problem it is assumed that crack propagation occurs along the plane of location of the crack. The case of the annular (round) macrocrack in a uniform body is examined in work [43]. Formulas obtained in these works for determining the value of limit stresses $p = p_*$ differ from formula (VII.24) by numerical factors.

On Fig. 71 are the change of $\frac{p_*}{\sigma_C}$ as a function of $\frac{a_*}{a}$ is graphed, the solid line using formulas (VII.23), and the dotted formula (VII.24). From comparison of graphs on this figure it follows that formulas (VII.23) and (VII.24) give practically identical values of the size of $\frac{p_*}{\sigma_0}$ if radius a of the considered crack satisfies the inequality

$$a > 10a_* = \frac{10\pi E \delta_k}{8(1-\nu^2)\sigma_0}. \qquad \text{(VII.25)}$$

Relationship (VII.25) can be used to evaluate the size of the minimum radius of a round macrocrack in the case of a three-dimensional problem.

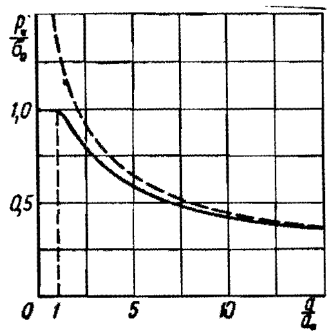

Fig. 71.

2. Discussion of Certain Theoretical and Experimental Data

Formula (VII.22) shows that in a real solid there can exist microscopic disk-shaped cracks which do not affect a decrease of the limit load during monotonic extension of a brittle body. The maximum diameter of these stable disk-shaped microcscopic cracks is determined on the basis of relationship (VII.23) by the following formula:

$$2a_* = \frac{\pi E \delta_k}{4(1-\nu^2)\sigma_0} = \frac{\pi E \gamma}{2(1-\nu^2)\sigma_0^2}. \qquad \text{(VII.26)}$$

Such "noninfluence" of small stress concentrators on the strength of a solid is observed, for example, for cast iron, since its brittle strength does not change when small stress concentrators are created in the body. The existance of nonpropagating, stable microscopic cracks was observed by I. A. Oding and B. S. Ivanov during a study of the fatigue strength of certain steels [107].

The conclusion concerning the existence of microscopic cracks which do not affect the static strength of a solid established within the bounds of a reference model of a brittle body, will agree with analogous conclusions of works [141, 148], made by authors also on the basis of power considerations, which can be formulated thus. Propagation of an internal microscopic (round) crack of diameter $2a < 2a_*$ for a deformable solid is strongly unadvantageous. This is because during local propagation of such a crack, the amount of liberated elastic energy, connected with the opening of the crack,

is less than the quantity of effective surface of the energy accumulating on its free surfaces.

A certain qualitative conformation of this phenomenon is apparently also found in investigations of L. A. Glikman [36], dedicated to the study of cyclical shock influence on brittle strength of steel during unilateral extension of specimens. In the mentioned work on cylindrical specimens made from carbon steel (curve 1 on Fig. 72) and chromium-nickel steel (curve 2) the following experiments were conducted. On the cylindrical surface of the samples was placed an external annular concentrator. Then to produce a uniform structure the samples were subjected to heat treatment.

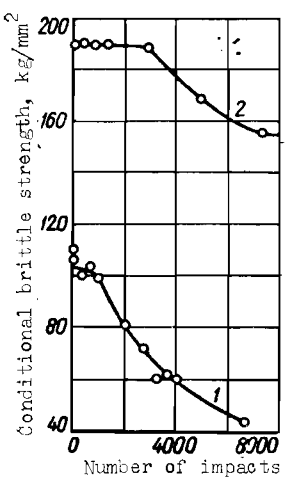

Fig. 72.

Subsequently the samples were subjected to deformation of extension by periodic-impact loads. After a certain number of such shocks the samples were subjected to static rupture at the temperature of liquid air. The purpose of the tests was to study the relative change of brittle strength due to shock fatigue of the material.

It is simple to note that experimental graphs on Fig. 72 are similar to theoretical graphs on Fig. 71. Moreover, according to experimental data, a certain low number of shock cycles of extension does not change the brittle strength of a given material. This, apparently, is explained by the fact that a low number of shock cycles leads to the development of only small (microscopic) cracks,

which according to the above considerations should (see Fig. 72) affect the brittle static strength of a sample.

We will conduct one more comparison of formula (VII.26). In the theory of rupture of solids advanced by Cottrell [183], in the structure of body is assumed the presence of initial microscopic cracks. The length of the biggest stable microscopic crack is determined by the equation

$$2\tilde{a}_* = \frac{E\gamma}{\pi(1-\nu^2)\sigma_0^2}, \qquad \text{(VII.27)}$$

where γ, σ_0, E and ν designate the same as in formula (VII.26).

Comparing formulas (VII.26) and (VII.27), we find that formula (VII.26) differs from formula (VII.27) only by the numerical factor $\frac{\pi^2}{2}$.

Work [192] publishes the results of low-temperature tests on the extension of samples made from two brands of steel and vacuum melt ferrite with different grain size. During the study of the formation of a microscopic crack in these materials, authors of work [192] came to the conclusion that the length of observable stable microscopic cracks in these materials is 1-2 grain diamters of the structure of the material. According to works [183, 192, 206] and according to results of calculation by formulas (VII.26) and (VII.27), if we consider that $E = 21\cdot10$ N/cm^2, $\gamma = 18\cdot10^3$ erg/cm^2, then in accordance with results of [183], it is possible to compose the following table:

σ_0, N/cm^2.....	$7.2\cdot10^4$ [183]	$5.3\cdot10^4$ [206]
Grain diameter, mm..........	0.025	0.11
$2\tilde{a}_*$, mm.......	0.0026	0.0049
$2a_*$, mm.......	0.012	0.022

These data show that the size of stable microscopic cracks calculated by the formula (VII.27) is an order less than the grain size of a corresponding structure, and the sizes of the same crack

calculated by formula (VII.26) are nearer in order of magnitude to the grain diameter of the structure of the material.

Considering this, it is possible to assume that formula (VII.23) and connected conclusions can be useful to explain certain phenomena of the mechanics of brittle rupture of solids.

3. The Case of Macroscopic Cracks in a Three-Dimensional Solid

Certain peculiarities of the brittle rupture of solids with microscopic disk-like cracks were examined above. In this section the example of a disk-like crack (see Fig. 70) will be used to examine characteristic peculiarities of the limit equilibrium state of three-dimensional brittle bodies with macroscopic (rather well-developed) cracks.

We assume that the examined round disk-like crack is macroscopic, i.e., characteristic linear dimension a of this crack is rather large. In this case (see Chapter I) it is possible to consider that

$$a_* \ll a, \quad R - a \ll a, \qquad \text{(VII.28)}$$

where a – radius of the circle limiting the macrocrack; R – radius of the contour of the microscopic crack, i.e., radius of the boundary between the region of weakened interpartial connections and the elastically deformed part of the body; a_* – quantity represented by formula (VII.19).

Note: If the contour of the examined disk-like macroscopic crack is not a circle, but is a certain closed smooth curve L, then in the neighborhood of point O_j on curve L radius a should be examined as the radius of curvature a_j of contour L in this point. Contour L is assumed smooth enough that condition (VII.28) holds.

For a macroscopic crack, as follows from formula (VII.24), the following inequality is characteristic also:

$$p_* \ll \sigma_0. \tag{VII.29}$$

For a macroscopic circular crack on the basis of formulas (VII.16) according to inequalities (VII.28) and (VII.29), we find

$$R - a = \frac{a}{2}\left(\frac{p}{\sigma_0}\right)^2 + \cdots, \tag{VII.30}$$

where the $0\left(\frac{p^4}{\sigma_0^4}\right)$ are omitted as small as compared to $\frac{p^2}{\sigma_0^2}$.

Formula (VII.30) is valid for an arbitrary value of montonically increasing load $p \leq p_*$. If load p reaches limit p_*, then, obviously, parameter R also attains a certain limit R_*. Consequently, on the basis of equality (VII.30) we determine the greatest end region of a macroscopic disk-like crack in a three-dimensional body (at the time of a maximum-equilibrium state of the crack) by the formula

$$R_* - a = \frac{a}{2}\left(\frac{p_*}{\sigma_0}\right)^2. \tag{VII.30a}$$

Hence and on the basis of formula (VII.24) we find

$$R_* - a = a_* = \frac{\pi E \delta_k}{8(1-\nu^2)\sigma_0}. \tag{VII.31}$$

Parameter a_* is constant for a given material. This parameter does not depend on the size of the crack and distribution of external loads, therefore from formula (VII.31) it follows that for a given material under assigned conditions (temperature, pressure, environment and so forth) the size of the end region in the neighborhood of points of the maximum-equilibrium state of the crack is constant. Let us also note that formula (VII.37) coincides with formula (I.46). Consequently, the size of the end region in the neighborhood of points of a maximum-equilibrium macroscopic crack for both the two-dimensional (plane deformation) and three-dimensional problem is identical.

Thus, formula (VII.31) shows that for macroscopic cracks in a three-dimensional body in the neighborhood of points of contour of crack in which the maximum-equilibrium state is attained, there exists autonomy of the end region of the crack. This circumstance will be used later in determining the limit load for a brittle three-dimensional body with a plane crack bound by a smooth curvilinear contour.

We will show now that if a three-dimensional brittle body with a plane macrocrack is subjected to extension by montonically increasing external loads, symmetric with respect to the plane of location of crack, then in this case the limit load should satisfy the equation:

$$\lim_{s_j \to 0} \{\sqrt{s_j}\,\sigma_z(s_j, 0, q_*)\} = \frac{K}{\pi}, \qquad \text{(VII.32)}$$

where K — modulus of cohesion; s_j — small distance between points of body located in plane of crack and the contour of the crack in the neighborhood of point O_j; $\sigma_z(s_j, 0, q_*)$ — elastic tensile stresses in plane of location of crack (plane z = 0) under load $q = q_*$.

At first we will examine the case when a brittle body is weakened by a macroscopic disk-like crack shaped like a circle with radius a. Let us assume that the sides of this crack are under montonically increasing axisymmetrical pressure $q(\rho)$.

Then, according to boundary conditions (VII.5), stresses $\sigma_z(r, 0, q)$ on the continuation of the crack can be represented so:

$$\sigma_z(r, 0, q) = \frac{2}{\pi\sqrt{r^2 - R^2}} \left\{ \int_0^R \frac{\rho\sqrt{R^2-\rho^2}}{r^2-\rho^2}\, q(\rho)\, d\rho - \right.$$

$$\left. - \sigma_0 \int_a^R \frac{\rho\sqrt{R^2-\rho^2}}{r^2-\rho^2}\, d\rho \right\},$$

where $r \geq R$.

Inasmuch as stressed $\sigma_z(r, 0, q)$ when $r \geq R$ must be finite, then, as noted above the following equality should hold:

$$\lim_{r \to R} \left\{ \int_0^R \frac{\rho \sqrt{R^2 - \rho^2}}{r^2 - \rho^2} q(\rho)\, d\rho - \sigma_0 \int_a^R \frac{\rho \sqrt{R^2 - \rho^2}}{r^2 - \rho^2} d\rho \right\} = 0.$$

Hence after calculation of the second integral we have

$$\lim_{r \to R} \left\{ \frac{\pi \sqrt{r^2 - R^2}}{2} \sigma_z(r, 0, q) \right\} = \sigma_0 \sqrt{R^2 - a^2}, \qquad \text{(VII.33)}$$

$\sigma_z(r, 0, q)$ are elastic tensile stresses appearing in points of the body located on a continuation of the crack (see Section 2, Chapter VI)

$$\sigma_z(r, 0, q) = \frac{2}{\pi \sqrt{r^2 - R^2}} \int_0^R \frac{\rho \sqrt{R^2 - \rho^2}}{r^2 - \rho^2} q(\rho)\, d\rho. \qquad \text{(VII.34)}$$

One should consider that the examined crack is macroscopic, i.e., the relationships

$$\varepsilon = \frac{R - a}{a} \ll 1; \quad R \approx a;$$
$$\sigma_0 (R - a) \neq 0.$$

do not hold for it.

Let us note relationship (VII.33) should hold for all values of $q \leq q_*$, and when $q = q_*$ this relationship takes the form

$$\lim_{s \to 0} \{\sqrt{s}\, \sigma_z(s, 0, q_*)\} = \frac{2\sigma_0}{\pi} \sqrt{R_* - a}, \qquad \text{(VII.35)}$$

where $s = r - a$.

The value of $R_* - a$ because of the autonomy of the end region of the macroscopic crack is a constant, therefore using formulas (VII.31) and (VII.35), we find

$$\lim_{s\to 0}\{\sqrt{s}\,\sigma_z(s,\ 0,\ q_*)\} = \frac{1}{\pi}\sqrt{\frac{4\sigma_0\pi E\delta_k}{8(1-\nu^2)}} \doteq \frac{K}{\pi}, \tag{VII.36}$$

QED for a macroscopic circular crack.

We will examine now the case when a plane macroscopic crack in a three-dimensional brittle body is bound by a smooth curvilinear contour L and the sides are under normal pressure q(x, y). We have on this contour arbitrary point O_j (Fig. 73). Part of arc $\overset{\frown}{AB}$ of contour L in a small neighborhood of point O_j can be examined as an arc of a circle of radius a_j. Consequently, taking into account formulas (VI.23) and (VI.30), elastic stresses $\sigma_z^+(s_j,\ 0,\ q)$ in the neighborhood of point O_j can be represented in the following form:

$$\sigma_z^+(s_j, 0, q) = \frac{N_j}{\sqrt{s_j}} + 0(1), \tag{VII.37}$$

where s_i – small distance between points of body located in the plane of the crack and point of the crack O_j; N_j – coefficient of intensity of elastic stresses at point O_j, which depends on the form of contour L and the form of the acting loads q.

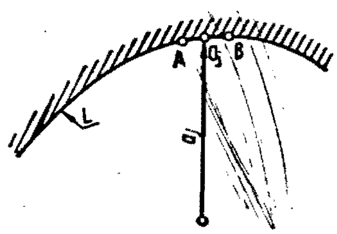

Fig. 73.

Inasmuch as the examined crack is macroscopic, characteristic linear dimension $R_j - a_j$ of the region of this crack, i.e., the region where cohesive forces σ_0 act, is considerably less than radius $a_j (R_j - a_j << a_j)$. This makes it possible to consider that stresses $\sigma_z^-(s_j,\ 0,\ \sigma_0)$ outside the cracks, induced by cohesive forces σ_0, coincide with stresses appearing in an infinite body with semi-infinite cut (see Fig. 17), on the surface of which is applied symmetric normal pressure $q(x) = -\sigma_0$ when $0 \leq x \leq R_j - a_j$. Considering this, and also on the basis of formula (I.76) we have

$$\sigma_z^-(s_j, 0, \sigma_0) = -\frac{2\sigma_0}{\pi\sqrt{s_j}}\sqrt{R_j - a_j} + 0(1). \qquad \text{(VII.38)}$$

The sum of stresses (VII.35) and (VII.36) should be finite in any point O_j of contour L. Therefore as mentioned above, the following equality should hold:

$$\lim_{s_j \to 0}\{\sqrt{s_j}\sigma_z^+(s_j, 0, q)\} = -\lim_{s_j \to 0}\{\sqrt{s_j}\sigma_z^-(s_j, 0, \sigma_0)\} = \frac{2\sigma_0}{\pi}\sqrt{R_j - a_j}.$$

If external load $q = q_*$ is such that in the neighborhood of certain point O_j parameter R_j reaches the limit $R = R_{j*}$, then in the neighborhood of this point the maximum-equilibrium state of the crack sets in, and, consequently, load $q = q_*$ becomes maximum.

Thus, from the last equality we see that external load $q = q_*$ will be maximum for given point O_j of the contour of the crack if

$$\lim_{s_j \to 0}\{\sqrt{s_j}\sigma_z(s_j, 0, q_*)\} = \frac{2\sigma_0}{\pi}\sqrt{R_{j*} - a_j}, \qquad \text{(VII.39)}$$

where $\sigma_z(s_j, 0, q_*)$ are elastic stresses in the neighborhood of point O_j of contour L when $q = q_*$.

For macroscopic cracks $R_{j*} - a_j$ is constant and is expressed by formula (VII.31). Putting in the right side of equation (VII.39) the value of $R_{j*} - a_j$ from formula (VII.31) and carrying out evident transformations, we arrive at equation (VII.32). Consequently, equation (VII.32) holds for a plane macroscopic crack bound arbitrary smooth curve L. We use this equation later in the study of the maximum equilibrium of three-dimensional brittle bodies with plane macrocracks.

4. Determination of the Limit Load for a Body Weakened by an External Circular Fracture

Let us examine an unlimited uniform brittle body weakened by a plane macrocrack which occupies all of plane z = 0 outside a circle

of radius a (Fig. 74). Let us refer this body to rectangular system of cartesian coordinates Oxyz so that plane xOy (z = 0) coincides with the plane of the crack, and the origin of coordinates with the center of the circle. Let us assume that the body is stretched along axis Oz by forces P, applied in points (0, 0, h) and (0, 0, - h). Let us determine the limit value of force $p = p_*$, after which an external circular crack starts to propagate over the cross section of the body in plane z = 0.

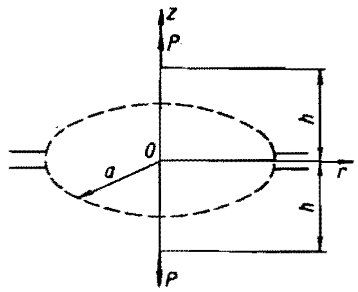

Fig. 74.

Because the crack is macroscopic, to determine the limit load $p = p_*$ we will use equation (VII.32). For this it is necessary to find the value of stresses $\sigma_z(r, 0)$ in points of the plane z = a when $r \leq a$. Preliminarily let us note that if in a three-dimensional body without a crack there act concentrated tensile forces P as is shown on Fig. 74, then in plane z = 0 of the body there appear tensile stresses [89, 90]

$$\sigma_z^{(1)}(r, 0) = \frac{P}{4\pi(1-\nu)} \cdot \frac{h}{(r^2+h^2)^{3/2}}\left[\frac{3h^2}{r^2+h^2} + 1 - 2\nu\right] \quad (0 \leqslant r \leqslant \infty). \qquad \text{(VII.40)}$$

By conditions of symmetry for our problem there are no tangential stresses on all of plane z = 0. In region $r \leq a$ of plane z = 0 normal shifts w(r, 0) = 0. These conditions permit using for determination of stresses $\sigma_z(r, 0, P)$ the solution of certain contact problems of the theory of elasticity for a half-space (see Section 2 Chapter VI) and to represent the unknown stresses as the sum of the following components:

$$\sigma_z(r, 0) = \sigma_z^{(1)}(r, 0) + \sigma_z^{(2)}(r, 0) + \sigma_z^{(3)}(r, 0), \qquad \text{(VII.41)}$$

where stresses $\sigma_z^{(1)}$, represented by formula (VII.40), are the stresses induced by external forces P in a body without a crack; $\sigma_z^{(2)}(r, 0)$ are the stresses appearing in region $0 \leq r \leq a$ of plane $z = 0$ of elastic half-space $z \geq 0$ as a result of the action of normla pressure $q(r) = \sigma_z^{(1)}(r, 0)$ outside this region and under the condition that points at infinity of the half-space are stationary; $\sigma_z^{(3)}(r, 0)$ are the stresses appearing and region $0 \leq r \leq a$ of plane $z = 0$ of elastic half-space $z \geq 0$, when in this region $w(r, 0) = 0$, and far from plane $z = 0$ tensile stresses $\sigma_z(r, h)$ are applied, the total vector of which throughout the plane $z = h (h \to \infty)$ is equal to D (in this case points at infinity of the body are not fixed).

Constant D for a given value of h is determined from the condition

$$\int_0^{\infty}\int_0^{2\pi} \sigma_z(r, 0)\, r\, d\alpha\, dr = P. \qquad \text{(VII.42)}$$

Stresses $\sigma_z^{(3)}(r, 0)$ are expressed so [33]:

$$\sigma_z^{(3)}(r, 0) = \frac{D}{2\pi a\sqrt{a^2 - r^2}} \quad (r < a), \qquad \text{(VII.43)}$$

and stresses $\sigma_z^{(2)}(r, 0)$ can be determined by the formula (VI.23), if we set in this formula $q(x, y) = q(r) = \sigma_z^{(1)}(r, 0)$.

In such a case

$$\sigma_z^{(2)}(r, 0) = \frac{1}{\pi^2\sqrt{a^2 - r^2}} \int_a^{\infty}\int_0^{2\pi} \frac{\rho\sqrt{\rho^2 - a^2}\,\sigma_z^{(1)}(\rho, 0)\, d\alpha\, d\rho}{r^2 + \rho^2 - 2r\rho\cos(\alpha - \beta)}, \qquad \text{(VII.44)}$$

where $r \leq a$.

In accordance with formula (VI.29) we find that the internal integral in equality (VII.44) has the form

$$\int_0^{2\pi} \frac{d\alpha}{r^2+\rho^2-2r\rho\cos(\alpha-\beta)} = \frac{2\pi}{\rho^2-r^2} \quad \text{when} \quad r<\rho.$$

On the basis of the last, formula (VII.44) can be written as:

$$\sigma_z^{(2)}(r, 0) = \frac{2}{\pi\sqrt{a^2-r^2}} \int_a^{\infty} \frac{\rho\sqrt{\rho^2-a^2}\,\sigma_z^{(1)}(\rho, 0)\,d\rho}{\rho^2-r^2} \qquad (r<a). \tag{VII.45}$$

Calculating by known formulas [39] the integral in formula (VII.45), and using then equalities (VII.40), (VII.41), we find

$$\begin{aligned}\sigma_z(r, 0) = \sigma_z^{(1)}(r, 0) &+ \frac{Ph}{2\pi^2(1-\nu)\sqrt{a^2-r^2}}\left\{\frac{1-2\nu}{h^2+r^2}+\right.\\ &+\frac{h^2(2h^2+3a^2-r^2)}{(a^2+h^2)(r^2+h^2)^2}-\\ &-\left.\frac{\sqrt{a^2-r^2}}{(r^2+h^2)^{3/2}}\left[1-2\nu+\frac{3h^2}{r^2+h^2}\right]\arcsin\sqrt{\frac{r^2+h^2}{a^2+h^2}}\right\}+\\ &+\frac{D}{2\pi a\sqrt{a^2-r^2}},\end{aligned} \tag{VII.46}$$

where $r \leq a$.

Placing expression (VII.46) in equality (VII.42) and carrying out the necessary calculations, we obtain

$$D = PD_0 = P\left[\frac{2}{\pi}\arcsin\frac{h}{\sqrt{h^2+a^2}} - \frac{ha}{\pi(1-\nu)(h^2+a^2)}\right]. \tag{VII.47}$$

By formulas (VII.40), (VII.46) and (VII.47) one can determine the unknown stresses $\sigma_z(r, 0)$ for our problem when $r \leq a$ (see Fig. 76).

Using further expressions (VII.46) and (VII.47), it is easy to find a formula for determination of the limit load $P = P_*$. This formula has the form

$$P_* = \frac{\pi(2a)^{3/2}(1-\nu)K(h^2+a^2)^2}{ah[3h^2+a^2-2\nu(h^2+a^2)]+\pi(1-\nu)(h^2+a^2)^2D_0}, \tag{VII.48}$$

where D_0 is determined from equality (VII.47).

When a point at infinity of the examined body is stationary, limit load P_* is determined by formula (VII.48) when $D_0 \equiv 0$ [114].

This formula can be written as:

$$P_* = \pi(1-\nu)(2a)^{3/2} K H(n) \qquad \left(n = \frac{h}{a}\right), \tag{VII.49}$$

where

$$H(n) = \\ = \frac{(1+n^2)^2}{n[1+3n^2-2\nu(1+n^2)]+(1+n^2)^2\left[2(1-\nu)\arcsin\frac{n}{\sqrt{1+n^2}}-\frac{n}{1+n^2}\right]}.$$

On Fig. 75 is the graph of the change of function H(n) when ν = 0.25.

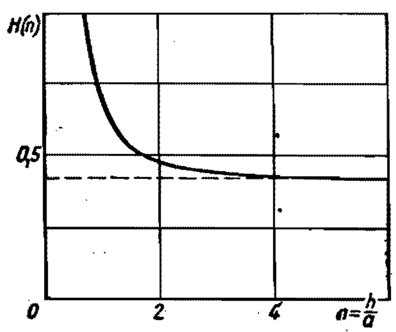

Fig. 75.

From formula (VII.49) it follows that as $n \to \infty$ ($h \to \infty$, a = const) for our problem (see Fig. 74) the limit load is determined by the formula

$$P_* = 2a\sqrt{2a}K. \tag{VII.50}$$

In this case forces $P = P_*$ must be examined as the main vector of external stresses applied to a body far from a crack.

C H A P T E R VIII

DETERMINATION OF LIMIT STRESSES FOR AN UNLIMITED BRITTLE SOLID HAVING A PLANE ELLIPSE-SHAPED FRACTURE

1. Formulation of the Problem

In a real three-dimensional brittle body cracks can have a random outline. This means that in the construction of a general theory of the rupture of brittle bodies determination of limit loads for a deformable brittle body weakened by cracks of random configuration is essential.

Let us consider a uniform unbounded brittle body, inside which is an elliptic (in the plane z = 0) macrascopic crack (Fig. 76). Let us introduce rectangular system of cartesian coordinates Oxyz so that axis Ox coincides with the major axis (2a), and axis Oy with the minor (2b). The crack is seen as ellipsoidal strips infinitely flattened along axis Oz.

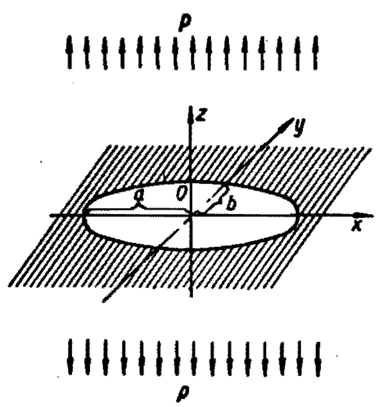

Fig. 76.

Let us assume that in points of infinity of the body montonically increasing tensile stresses p are applied, directed along axis $Oz[\sigma_z(x, y, \infty) = p]$. It is required to determine limit stresses $p = p_*$.

The state of strain in a body with an elliptic crack is represented in the form of the sum of: 1) the state of strain in a body without a crack, when in points at infinity there act tensile stresses $\sigma_z(x, y, \infty) = p$, and 2) the state of strain in a body with an elliptic crack, when the surface of this crack normal pressure $q(x, y) = p$ is applied, and in points at infinity of the body there are not stresses.

The first state of strain is characterized by only one component of the stress tensor not equal to zero.

$$\sigma_z^{(0)}(x, y, z) = p. \qquad \text{(VIII.1)}$$

For the second state of strain plane z = 0 is the plane symmetry. This means determination of stress is and deformation in the examined body (Fig. 76) lead to resolution of a problem of the theory of elasticity for an elastic half-space $z \geq 0$, when on its surface (z = 0) the following boundary conditions are assigned:

$$\begin{aligned} &\tau_{xz} = \tau_{yz} = 0 && \text{when} \quad z = 0; \\ &\sigma_z^{(1)}(x, y, 0) = -p && \text{when} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1; \\ &w(x, y, 0) = 0 && \text{when} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \geqslant 1, \end{aligned} \qquad \text{(VIII.2)}$$

where τ_{xz}, τ_{yz}, $\sigma_z^{(1)}$ are components of the elastic stress tensor for the second stressed state; w(x, y, 0) — component of the vector of shifts along axis Oz.

On the basis of boundary conditions (VII.2) one can determine the vertical shift of the surface of an elastic half-space, i.e., function

$$w(x, y, 0) \text{ when } \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1,$$

and stress component

$$\sigma_z^{(1)}(x, y, 0) \text{ when } \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1.$$

If this problem will be solved, then stresses $\sigma_z(x, y, 0)$ for our body with a plane elliptic crack (Fig. 76) can be represented (for points located outside the region of the crack) as:

$$\sigma_z(x, y, 0) = \sigma_z^0(x, y, 0) + \sigma_z^{(1)}(x, y, 0). \qquad \text{(VIII.3)}$$

On the basis of this formula and equation (VII.32) one can determine limit stresses $p = p_*$.

2. Determination of Tensile Stresses in the Plane of an Elliptic Crack

To determine stresses $\sigma_z^{(1)}(x, y, 0)$ and shifts $w(x, y, 0)$ in the plane of an elliptic crack (Fig. 76) we will use equation (VI.20), which according to boundary conditions (VIII.2) takes the following form:

$$\sigma_z^{(1)}(x, y, 0) = = \frac{E}{4\pi(1-\nu^2)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \iint\limits_{\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} < 1\right)} \frac{w(\xi, \eta, 0)\, d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}. \qquad \text{(VIII.4)}$$

For points $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, $z = 0$ the value of stresses $\sigma_z(x, y, 0)$ is known ($\sigma_z = -p$), consequently, in order to determine function $w(x, y, 0)$ when $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, from formula (VIII.4) we obtain the equation

$$p = \frac{-E}{4\pi(1-\nu^2)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \iint\limits_{\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} < 1\right)} \frac{w(\xi, \eta, 0)\, d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} \qquad \text{(VIII.5)}$$

when $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$

We seek a continuous solution to equation (VIII.5), turning into zero on the contour of the examined region by analogy with a circular crack in the following form:

$$w(x, y, 0) = c_0 \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}, \qquad \text{(VIII.6)}$$

where c_0 is a constant which can be determined from equation (VIII.5).

Placing expression (VIII.6) in equation (VIII.5) and carrying out the necessary integration, and then differentiation, we obtain [110]

$$c_0 = \frac{2(1-\nu^2)\,bp}{EE(k)}. \qquad \text{(VIII.7)}$$

Here E(k) is a full elliptic integral of the second kind,

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \alpha}\, d\alpha,$$

where

$$k^2 = \frac{a^2 - b^2}{a^2}.$$

Thus, for our problem (Fig. 76) shifts w(x, y, 0), determinate in all of plane z = 0, are expressed by formula

$$w(x, y, 0) = \begin{cases} 0 & \text{when } \frac{x^2}{a^2} + \frac{y^2}{b^2} \geqslant 1; \\ c_0 \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} & \text{when } \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1. \end{cases} \qquad \text{(VIII.8)}$$

Now, using expressions (VIII.4) and (VIII.8), for determination of stresses $\sigma_z(x, y, 0)$ when $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ we obtain the expression

$$\sigma_z(x, y, 0) = \frac{c_0 E}{4\pi(1-\nu^2)} \Delta_{xy} f(x, y), \tag{VIII.9}$$

where

$$\Delta_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2};$$

$$f(x, y) = \iint\limits_{\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}<1\right)} \frac{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\, d\xi d\eta. \tag{VIII.9a}$$

After calculation of function f(x, y) and transformations in accordance with formula (VIII.9), we find the desired stresses $\sigma_z(x, y, 0)$.

Function f(x, y) is calculated in the following way. First we make the following consideration. Let us assume that continuous function h(x, y) is assigned in all of plane xOy. Then integral

$$\iint\limits_{(\infty)} \frac{h(\xi, \eta)\, d\xi d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}} = Z[h(x, y)] \tag{VIII.10}$$

represents a certain function of variables x and y in this plane. To calculate function Z[h(x, y)] we introduce polar coordinates (α, t), assuming that

$$\xi = x + t\cos\alpha; \quad \eta = y + t\sin\alpha, \tag{VIII.11}$$

when

$$t = \sqrt{(x-\xi)^2+(y-\eta)^2}.$$

In this case expression (VIII.10) obtains the form

$$Z[h(x, y)] = \int\limits_0^\pi \int\limits_{-\infty}^{\infty} h[t, \alpha]\, dt d\alpha, \tag{VIII.12}$$

where

$$h[t, \alpha] = h[\xi(t, \alpha), \eta(t, \alpha)].$$

In formula (VIII.12) the internal integral

$$s(\alpha) = \int_{-\infty}^{\infty} h[t, \alpha]\, dt, \qquad \text{(VIII.13)}$$

taken along line α = const, constitutes the area $s(\alpha)$ of the cross section of a certain body bound by plane xOy and surface $h[t, \alpha]$. At fixed values of parameters x, y area $s(\alpha)$ is a function of only angle α. Thus, formula (VIII.12) can be represented in the following form:

$$Z[h(x, y)] = \int_0^{\pi} s(\alpha)\, d\alpha, \qquad \text{(VIII.14)}$$

where $s(\alpha)$ is determined by the formula (VIII.13).

Let us examine now the case when

$$h(x, y) = \begin{cases} \sqrt{1 - \frac{x^2}{b^2} - \frac{y^2}{b^2}} & \text{when } \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1; \\ 0 & \text{when } \frac{x^2}{a^2} + \frac{y^2}{b^2} \geqslant 1. \end{cases} \qquad \text{(VIII.15)}$$

If expression (VIII.15) is placed in (VIII.10), then we obtain the following:

$$f(x, y) = Z[h(x, y)].$$

We go now to calculation of functions $f(x, y)$. In system of rectangular cartesian coordinates x, y, z function (VIII.15) represents the surface of half of an ellipsoid, located above plane z = 0. Cross sections of such an ellipsoid of arbitrary plane perpendicular to coordinate plane z = 0 will be ellipses. To determine semiaxes of ellipses we go in formula (VIII.15) to polar coordinates (VIII.11). Then

$$h[t, \alpha] = \sqrt{1 - \frac{(x + t\cos\alpha)^2}{a^2} - \frac{(y + t\sin\alpha)^2}{b^2}}. \qquad \text{(VIII.16)}$$

After simple transformations expression (VIII.16) can be written as:

$$h[t, \alpha] = b^* \sqrt{1 - \left(\frac{t + c^*}{a^*}\right)^2}, \qquad \text{(VIII.17)}$$

where

$$b^* = \sqrt{1 - \frac{(x \sin\alpha - y \cos\alpha)^2}{a^2 \sin^2\alpha + b^2 \cos^2\alpha}}; \quad a^* = \frac{b^* ab}{\sqrt{a^2 \sin^2\alpha + b^2 \cos^2\alpha}};$$
$$c^* = \frac{b^2 x \cos\alpha + a^2 y \sin\alpha}{a^2 \sin^2\alpha + b^2 \cos^2\alpha}. \qquad \text{(VIII.18)}$$

Consequently, for fixed values of x, y, α equation (VII.17) represents an arc of an ellipse whose semiaxes a^* and b^* can be determined by formulas (VIII.18). Area $s(\alpha)$ of such an semiellipse is expressed by the formula

$$s(\alpha) = \frac{\pi a^* b^*}{2} = \frac{\pi ab}{2}\left[1 - \frac{(x \sin\alpha - y \cos\alpha)^2}{a^2 \sin^2\alpha + b^2 \cos^2\alpha}\right] \times$$
$$\times \frac{1}{\sqrt{a^2 \sin^2\alpha + b^2 \cos^2\alpha}}. \qquad \text{(VIII.19)}$$

At fixed values of x and y area $s(\alpha)$ is a function of angle α. It is necessary to consider that function (VIII.19) has meaning when angle α (Fig. 77) satisfies the conditions

$$0 \leqslant \alpha \leqslant \pi \quad \text{when } \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1; \qquad \text{(VIII.20)}$$
$$\alpha_1 \leqslant \alpha \leqslant \alpha_2 \quad \text{when } \frac{x^2}{a^2} + \frac{y^2}{b^2} \geqslant 1. \qquad \text{(VIII.20a)}$$

If values of angle α do not satisfy condition (VIII.20a), when $\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1$, then function $h[t, \alpha]$, represented by formula (VIII.15), is identically equal to zero and, consequently, area $s(\alpha)$ also is equal to zero.

Angles α_1 and α_2, appearing in condition (VIII.21), are the angles between $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, passing through certain point M(x, y) when $\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1$, and axis Ox (Fig. 77). Taking this into account we find

$$\alpha_1(x, y) = \\ = \operatorname{Arctg} \frac{-b^2x + y\sqrt{a^2y^2 + b^2x^2 - a^2b^2}}{a^2y + x\sqrt{a^2y^2 + b^2x^2 - a^2b^2}}; \quad \text{(VIII.21)}$$

$$\alpha_2(x, y) = \operatorname{Arctg} \frac{b^2x + y\sqrt{a^2y^2 + b^2x^2 - a^2b^2}}{-a^2y + x\sqrt{a^2y^2 + b^2x^2 - a^2b^2}}. \quad \text{(VIII.21a)}$$

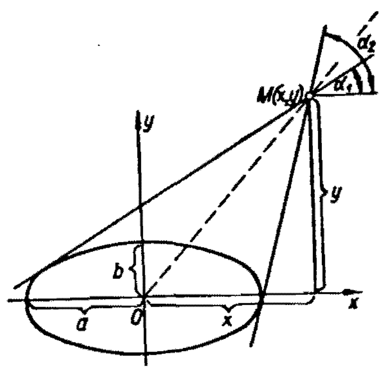

Fig. 77.

Formulas (VIII.21) and (VIII.21a) can also be transformed to the form (see Fig. 77)

$$\alpha_1(x, y) = \operatorname{Arctg} \frac{y}{x} - \\ - \operatorname{arctg} \frac{a^2y^2 + b^2x^2}{(x^2 + y^2)\sqrt{a^2y^2 + b^2x^2 - a^2b^2} + xy(a^2 - b^2)}; \quad \text{(VIII.22)}$$

$$\alpha_2(x, y) = \operatorname{Arctg} \frac{y}{x} + \\ + \operatorname{arctg} \frac{a^2y^2 + b^2x^2}{(x^2 + y^2)\sqrt{a^2y^2 + b^2x^2 - a^2b^2} - xy(a^2 - b^2)}. \quad \text{(VIII.23)}$$

Thus, on the basis of formulas (VIII.14), (VIII.15) and (VIII.19) we obtain

$$f(x, y) = \frac{\pi ab}{2} \int_{\alpha_1(x, y)}^{\alpha_2(x, y)} \left[1 - \frac{(x \sin \alpha - y \cos \alpha)^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}\right] \times \\ \times \frac{d\alpha}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}}, \quad \text{(VIII.24)}$$

where parameters $\alpha_1(x, y)$ and $\alpha_2(x, y)$ are represented by equalities (VIII.22) and (VIII.23) when

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1.$$

Considering in expression (VIII.24)

$$\alpha = \frac{\pi}{2} + \Theta \text{ and } k^2 = \frac{a^2 - b^2}{a^2}$$

and carrying out simple transformations, we find

$$f(x, y) = \pi b \left\{ \frac{1}{2} \int_{\Theta_1}^{\Theta_2} \frac{d\Theta}{\sqrt{1 - k^2 \sin^2 \Theta}} - \frac{1}{2} \cdot \frac{x^2}{a^2} \int_{\Theta_1}^{\Theta_2} \frac{\cos^2 \Theta \, d\Theta}{(1 - k^2 \sin^2 \Theta)^{3/2}} - \right.$$
$$\left. - \frac{xy}{a^2} \int_{\Theta_1}^{\Theta_2} \frac{\sin \Theta \cos \Theta \, d\Theta}{(1 - k^2 \sin^2 \Theta)^{3/2}} - \frac{1}{2} \cdot \frac{y^2}{a^2} \int_{\Theta_1}^{\Theta_2} \frac{\sin^2 \Theta \, d\Theta}{(1 - k^2 \sin^2 \Theta)^{3/2}} \right\}, \qquad \text{(VIII.25)}$$

where

$$\Theta_1 = \Theta_1(x, y) = \alpha_1(x, y) - \frac{\pi}{2};$$
$$\Theta_2 = \Theta_2(x, y) = \alpha_2(x, y) - \frac{\pi}{2}. \qquad \text{(VIII.26)}$$

Let us now turn to calculation of stresses. In accordance with formulas (VIII.9) and (VIII.25) we determine stresses $\sigma_z^{(1)}(x, y, 0)$ when $\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1$. Using these formulas and subjecting function $f(x, y)$ to the operation Δ_{xy}, we obtain

$$\Delta_{xy} f(x, y) = \omega(x, y) = \pi b \left\{ - \frac{1}{a^2} \int_{\Theta_1}^{\Theta_2} \frac{d\Theta}{(1 - k^2 \sin^2 \Theta)^{3/2}} + \right.$$
$$+ \frac{2 \sin^2 \Theta_1 (x \cos \Theta_1 + y \sin \Theta_1)}{a^2 (1 - k^2 \sin^2 \Theta_1)^{3/2}} \left(\frac{\partial S_1}{\partial x} \cos \Theta_1 + \frac{\partial S_1}{\partial y} \sin \Theta_1 \right) -$$
$$- \frac{2 \sin^2 \Theta_2 (x \cos \Theta_2 + y \sin \Theta_2)}{a^2 (1 - k^2 \sin^2 \Theta_2)^{3/2}} \left(\frac{\partial S_2}{\partial x} \cos \Theta_2 + \frac{\partial S_2}{\partial y} \sin \Theta_2 \right) +$$
$$+ \frac{\sin^3 \Theta_1}{2a^2 (1 - k^2 \sin^2 \Theta_1)^{5/2}} [-a^2 (1 - k^2 \sin^2 \Theta_1)(2 - k^2 \sin^2 \Theta_1) \cos \Theta_1 + \qquad \text{(VIII.27)}$$

$$+ x^2 \cos\Theta_1 (2\cos 2\Theta_1 + k^2 \sin^2\Theta_1 + k^2 \sin^4\Theta_1) +$$
$$+ 2xy \sin\Theta_1 (3\cos^2\Theta_1 - \sin^2\Theta_1 + k^2\sin^4\Theta_1) + y^2 \sin^2\Theta_1 \cos\Theta_1 \times$$
$$\times (4 - k^2\sin^2\Theta_1)\Big]\left[\left(\frac{\partial S_1}{\partial x}\right)^2 + \left(\frac{\partial S_1}{\partial y}\right)^2\right] + \frac{\sin^3\Theta_2}{2a^2(1-k^2\sin^2\Theta_2)^{3/2}}\,[a^2(1 -$$
$$- k^2\sin^2\Theta_2)(2 - k^2\sin^2\Theta_2)\cos\Theta_2 - x^2\cos\Theta_2(2\cos 2\Theta_2 + k^2\sin^2\Theta_2 +$$
$$+ k^2\sin^4\Theta_2) - 2xy\sin\Theta_2(3\cos^2\Theta_2 - \sin^2\Theta_2 + k^2\sin^4\Theta_2) -$$
$$- y^2\sin^2\Theta_2\cos\Theta_2(4 - k^2\sin^2\Theta_2)\Big]\left[\left(\frac{\partial S_2}{\partial x}\right)^2 + \left(\frac{\partial S_2}{\partial y}\right)^2\right]\Big\}, \qquad \text{(VIII.27 Cont'd.)}$$

where

$$S_1 = \frac{-b^2x + y\sqrt{a^2y^2 + b^2x^2 - a^2b^2}}{a^2y + x\sqrt{a^2y^2 + b^2x^2 - a^2b^2}};$$
$$S_2 = \frac{b^2x + y\sqrt{a^2y^2 + b^2x^2 - a^2b^2}}{-a^2y + x\sqrt{a^2y^2 + b^2x^2 - a^2b^2}};$$
$$\Theta_1 = \operatorname{Arctg} S_1 - \frac{\pi}{2}; \quad \Theta_2 = \operatorname{Arctg} S_2 - \frac{\pi}{2}. \qquad \text{(VIII.27a)}$$

If we place expression (VIII.27a) in (VIII.27), then after transformations we will obtain

$$\omega(x, y) = \pi b\left\{-\frac{1}{a^2}\int_{\Theta_1}^{\Theta_2}\frac{d\Theta}{(1-k^2\sin^2\Theta)^{3/2}} +\right.$$
$$+ \frac{2a\left[\sqrt{g_2^2+g_3^2}+\sqrt{g_4^2+g_5^2}\right]}{(a^2y^2+b^2x^2)\sqrt{g_1}} - \frac{a}{2(a^2y^2+b^2x^2)^3 g_1}\times$$
$$\left.\times\left[\frac{N_1(x, y)}{g_2\sqrt{g_2^2+g_3^2}} + \frac{N_2(x, y)}{g_5\sqrt{g_4^2+g_5^2}}\right]\right\}, \qquad \text{(VIII.28)}$$

where

$$g_1 = a^2y^2 + b^2x^2 - a^2b^2; \quad g_2 = a^2y + x\sqrt{g_1};$$
$$g_3 = -b^2x + y\sqrt{g_1}; \quad g_4 = b^2x + y\sqrt{g_1}; \quad g_5 = -a^2y + x\sqrt{g_1};$$
$$N_1(x, y) = -g_3(a^2y^2 + b^2x^2)^2[(a^2 + b^2)g_2^2 + 2a^2g_3^2] +$$
$$+ x^2a^2g_3(2g_3^4 - 2g_2^4 + 2k^2g_2^4 + k^2g_2^2g_3^2) -$$
$$- 2xya^2g_2(2g_2^2g_3^2 + 3g_3^4 - g_2^4 + k^2g_2^4) +$$
$$+ y^2a^2g_2^2g_3(4g_2^2 + 4g_3^2 - k^2g_2^2); \qquad \text{(VIII.28a)}$$
$$N_2(x, y) = g_4(a^2y^2 + b^2x^2)^2[(a^2 + b^2)g_5^2 + 2a^2g_4^2] -$$
$$- x^2a^2g_4(2g_4^4 - 2g_5^4 + 2k^2g_5^4 + k^2g_4^2g_5^2) +$$
$$+ 2xya^2g_5(2g_4^2g_5^2 + 3g_4^4 - g_5^4 + k^2g_5^4) -$$
$$- y^2a^2g_4g_5^2(4g_5^2 + 4g_4^2 - k^2g_5^2);$$
$$k^2 = \frac{a^2 - b^2}{a^2}.$$

Thus, using relationships (VIII.7), (VIII.9), (VIII.27), (VIII.28) to determine the stresses we obtain the expression

$$\sigma_z^{(1)}(x, y, 0) = \begin{cases} -p & \text{when } \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1; \\ \frac{bp}{2\pi E(k)}\omega(x, y) & \text{when } \frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1. \end{cases} \quad \text{(VIII.29)}$$

Based on this formula, and also equalities (VIII.2) and (VIII.3) for determination of elastic stresses $\sigma_z(x, y, 0)$ in a brittle body with a plane elliptic crack (see Fig. 76), when the body is subjected to uniform extension in infinitely remote its points by external stresses $\sigma_z(x, y, \infty) = p$, we have the following formula:

$$\sigma_z(x, y, 0) = \begin{cases} 0 & \text{when } \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1; \\ p + \frac{bp}{2\pi E(k)}\omega(x, y) & \text{when } \frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1, \end{cases} \quad \text{(VIII.30)}$$

where function $\omega(x, y)$ is expressed by equalities (VIII.27) or (VIII.28).

Note: Let us note that the problem about distribution of stresses in an elastic body with an elliptic crack was examined also in works [64, 189]. Moreover in work [64] this problem is reduced to finding a harmonic function in an elastic half-space, and in [189] the solution is given in almost unacceptable form for calculating stresses in the neighborhood of the contour of the examined crack [200]. The solution given here is given in work [116].

Let us examine certain particular cases of solving the problem.

1. Using formula (VIII.30), we will determine for example, for the problem represented on Fig. 76, the distribution of stresses $\sigma_z(x, y, 0)$ along axis Oy when $y \geq b$, i.e., in points of plane $z = 0$ when $x = 0$ and $y \geq b$. In this case on the basis of formula (VIII.22), (VIII.23) and (VIII.26)

$$\Theta_1^* = \Theta_1(0,\ y) = -\frac{\pi}{2} + \operatorname{arctg}\frac{\sqrt{y^2-b^2}}{a},$$

$$\Theta_2^* = \Theta_2(0,\ y) = \frac{\pi}{2} - \operatorname{arctg}\frac{\sqrt{y^2-b^2}}{a}. \qquad \text{(VIII.31)}$$

When x = 0, y $\geq$ b formula (VIII.28) takes the form

$$\omega(0,\ y) = \pi b\left\{-\frac{1}{a^2}\int\limits_{\Theta_1^*}^{\Theta_2^*}\frac{d\Theta}{(1-k^2\sin^2\Theta)^{3/2}} + \frac{2\sqrt{y^2+a^2-b^2}}{ay\sqrt{y^2-b^2}}\right\}. \qquad \text{(VIII.32)}$$

The integral in formula (VIII.32), according to relationships (VIII.31), after certain transformations can be written as:

$$\int\limits_{\Theta_1^*}^{\Theta_2^*}\frac{d\Theta}{(1-k^2\sin^2\Theta)^{3/2}} = 2\frac{a^2}{b^2}\left[E(k) - \int\limits_0^{\psi_1}\sqrt{1-k^2\sin^2\psi}\,d\psi\right], \qquad \text{(VIII.33)}$$

where

$$\psi_1 = \operatorname{arctg}\frac{\sqrt{y^2-b^2}}{b};\ k^2 = \frac{a^2-b^2}{a^2}. \qquad \text{(VIII.34)}$$

Using further formulas (VIII.30), (VIII.32) and (VIII.33), we find

$$\sigma_z(0,\ y,\ 0) = \frac{p}{E(k)}\left\{\int\limits_0^{\psi_1}\sqrt{1-k^2\sin^2\psi}\,d\psi + \frac{b^2\sqrt{y^2+a^2-b^2}}{ay\sqrt{y^2-b^2}}\right\}, \qquad \text{(VIII.35)}$$

where y $\geq$ b; values of ψ_1 and k are claculated by formulas (VIII.34).

If in formula (VIII.35) we set a $\to \infty$(k $\to$ 1, b = const) and note that E(1) = 1, then as a result of simple transformations we will obtain

$$[\sigma_z(0,\ y,\ 0)]_{a\to\infty} = p\left(\sin\psi_1 + \frac{b^2}{y\sqrt{y^2-b^2}}\right).$$

Hence and on the basis of relationships (VIII.34) we find the formula for determination of stresses in an infinite plate weakened

by a linear crack and subjected to uniaxial extension. This formula (see also formula (I.1)) has the form

$$[\sigma_z(0,\ y,\ 0)]_{a=\infty} = \frac{py}{\sqrt{y^2 - b^2}} \quad (y \geqslant b).$$

2. Let us determine the stresses $\sigma_z(x, y, 0)$ when $y = 0$ and $x \geq a$. For points of plane $z = 0$ (see Fig. 76) under these conditions functions Θ_1 and Θ_2 according to the formula (VIII.22), (VIII.23) and (VIII.26) take the form

$$\begin{aligned} \Theta_1^{**} &= \Theta_1(x,\ 0) = -\frac{\pi}{2} - \operatorname{arctg}\frac{b}{\sqrt{x^2 - a^2}}; \\ \Theta_2^{**} &= \Theta_2(x,\ 0) = -\frac{\pi}{2} + \operatorname{arctg}\frac{b}{\sqrt{x^2 - a^2}}. \end{aligned} \qquad \text{(VIII.36)}$$

When $y = 0$, $x \geq a$ formula (VIII.28) after transformations can be written as:

$$\omega(x,\ 0) = \pi b \left\{ -\frac{1}{a^2} \int_{\Theta_1^{**}}^{\Theta_2^{**}} \frac{d\Theta}{(1 - k^2 \sin^2 \Theta)^{3/2}} - \frac{2a}{b^2 x} \cdot \frac{\sqrt{x^2 + b^2 - a^2}}{\sqrt{x^2 - a^2}} \right\}. \qquad \text{(VIII.37)}$$

The integral in this formula is expressed through an elliptic function by the equality:

$$\int_{\Theta_1^{**}}^{\Theta_2^{**}} \frac{d\Theta}{(1 - k^2 \sin^2 \Theta)^{3/2}} = 2\frac{a^2}{b^2} \int_0^{\psi_2} \sqrt{1 - k^2 \sin^2 \psi}\, d\psi, \qquad \text{(VIII.38)}$$

where

$$\psi_2 = \operatorname{arctg}\frac{a}{\sqrt{x^2 - a^2}} \quad (x > a).$$

Consequently, according to formula (VIII.30), (VIII.37) and (VIII.38)

$$\sigma_z(x,\ 0,\ 0) = p + \frac{p}{E(k)} \left\{ \frac{a}{x} \cdot \frac{\sqrt{x^2 + b^2 - a^2}}{\sqrt{x^2 - a^2}} - \int_0^{\psi_2} \sqrt{1 - k^2 \sin^2 \psi}\, d\psi \right\}, \qquad \text{(VIII.39)}$$

where

$$x \geqslant a.$$

3. Let us consider the case when a brittle three-dimensional isotropic body contains a plane round macrocrack, and the body is extended in infinitely remote points by constant stresses p, directed perpendicularly to the plane of the crack. For this problem stresses $\sigma_z^{(1)}(x, y, 0)$ can be determined on the basis of formulas (VIII.28) and (VIII.30), if we set a = b, where a is the radius of the examined crack.

In this case from formulas (VIII.22), (VIII.23) and (VIII.26) follow the equalities

$$\begin{aligned} \Theta_1^{(0)}(x, y) &= -\frac{\pi}{2} + \arcsin\frac{y}{\sqrt{x^2+y^2}} - \arcsin\frac{a}{\sqrt{x^2+y^2}}; \\ \Theta_2^{(0)}(x, y) &= -\frac{\pi}{2} + \arcsin\frac{y}{\sqrt{x^2+y^2}} + \arcsin\frac{a}{\sqrt{x^2+y^2}}. \end{aligned} \qquad \text{(VIII.40)}$$

Furthermore, when a = b we have

$$k = 0, \quad [E(k)]_{k=0} = \frac{\pi}{2}. \qquad \text{(VIII.41)}$$

Assuming that in equality (VIII.28) a = b and using formulas (VIII.40) and (VIII.41), we find

$$[\omega(x, y)]_{a=b} = \frac{2\pi}{a}\left(\frac{a}{\sqrt{r^2-a^2}} - \arcsin\frac{a}{r}\right), \qquad \text{(VIII.42)}$$

where $x^2 + y^2 = r^2 \geq a^2$.

Hence and on the basis of formula (VIII.30) and equality (VIII.41) when a = b and $x^2 + y^2 = r^2 \geq a^2$ we obtain the formula for determination of stresses

$$\sigma_z(r, 0) = p + \frac{2}{\pi} p\left(\frac{a}{\sqrt{r^2-a^2}} - \arcsin\frac{a}{r}\right). \qquad \text{(VIII.43)}$$

This formula agrees with the result in work [63].

3. Determination of Limit Stresses

Let us now turn to determination of the limit stresses $p = p_*$ for a plane elliptic macrocrack contained inside a brittle body subjected to uniaxial extension by monotonically increasing stresses $\sigma_z(x, y, \infty) = p$ (see Fig. 76). Inasmuch as the crack is macroscopic, then to calculate stresses $p = p_*$ it is possible to use equation (VII.32).

Using this equation and results of the preceding section we find the limit stresses $p = p_*$ after which the propagation of a plane elliptic crack begins in the direction of the minor or major axis.

For the minor axis of an elliptic crack we have $x = 0$, $s_j = y - b$, and elastic rupture stresses $\sigma_z(o, y, 0)$ are determined by formula (VIII.35). Using this formula and condition (VIII.32) for determination of the limit load $p = p_*(b)$ after which an elliptic crack starts to propagate in the direction of its minor axis, we have the equation

$$\lim_{y \to b} \left\{ \frac{p_*^{(b)}}{E(k)} \sqrt{y - b} \left[\int_0^{\psi_1} \sqrt{1 - k^2 \sin^2 \psi}\, d\psi + \frac{b^2}{ay} \times \right.\right.$$
$$\left.\left. \times \frac{\sqrt{y^2 + a^2 - b^2}}{\sqrt{y^2 - b^3}} \right] \right\} = \frac{K}{\pi}.$$

Carrying out in this equation passage to the limit as $y \to b$, we find

$$p_*^{(b)} = E(k) \frac{\sqrt{2}\, K}{\pi \sqrt{b}} = E(k) \sqrt{\frac{2E\gamma}{\pi (1 - \nu^2) b}} \qquad \text{(VIII.44)}$$

It is easy to note that when $a = \infty$ ($k = 1$, $E(1) = 1$) from this formula follows the Griffith formula (see Chapter I) for a plate with a linear crack whose half-length is $l = b$.

Formula (VIII.44) is set by the above means in a work of the author [116]; then it was found on the basis of the theory of macrostresses in a work of M. Ya. Leonov and K. N. Rusinko [76], and also obtained by other means by Irwin in work [200].

For the major axis of an elliptic crack we have $y = 0$, $s_j = x - a$. Thus, on the basis of equation (VII.32) and formula (VIII.39) we find.

$$p_*^{(a)} = \sqrt{\frac{a}{b}}\, E(k) \frac{\sqrt{2}\, K}{\pi \sqrt{b}}. \qquad \text{(VIII.45)}$$

If in formulas (VIII.44) or (VIII.45) we set $a = b$, then as a special case we obtain the Saha formula [182]:

$$p_*^{(a)} = p_*^{(b)} = \frac{K}{\sqrt{2b}} \quad (a = b).$$

Obviously, this formula can be obtained directly on the basis of equation (VII.32) and formula (VIII.43).

Using formulas (VIII.44) and (VIII.45), it is possible to graph the change of limit load $\frac{p_*}{\sigma_k}$ ($\sigma_k = K/\sqrt{2a}$) depending upon the relationship between strips ($k^1 = b/a$) if the elliptic crack. Such graphs are represented on Fig. 78, where curve 1 corresponds to calculations by formula (VIII.44), and curve 2 by formula (VIII.45).

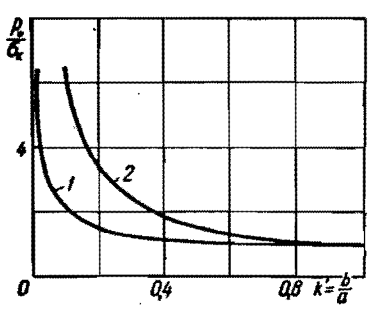

Fig. 78.

On the basis of graphs on Fig. 78 it is easy to note that $p_*^{(b)} \leq p_*^{(a)}$ when $b \leq a$. This means that a plane elliptic crack contained inside a brittle isotropic body subjected to uniaxial extension by monotonically increasing stresses $\sigma_z(x, y, \infty) = p$ directed perpendicularly to the plane of the crack at first propagates in the direction of the minor axis of the crack, i.e., tends toward formation of a round crack.

In conclusion let us note that load $p_*^{(b)}$ at which the initial elliptic crack becomes moving-equilibrium, i.e., starts to propagate in the direction of its minor axis, is not generally a rupture load for a body with such a crack. In this case, in order to establish the level of the rupture load it is necessary to study the kinetics of an elliptic crack, i.e., it is necessary to find the limit load for the crack obtained from a plane elliptic crack after external load p reaches $p_*^{(b)} + \Delta p$. An approximate analysis of this problem is in Chapter IX.

C H A P T E R IX

PROPAGATION OF AN ALMOST CIRCULAR CRACK

1. Formulation of the Problem

We will examine an isotropic unbounded brittle body, assuming that in this body is a plane isolated macrocrack (Fig. 79). Designating by S_0 the region occupied by the crack, and by L_0 the contour of this region, we will introduce rectangular system of cartesian coordinates Oxyz in such a way that the plane of the crack S_0 coincides with plane xOy, and the origin of coordinates coincides with the center of the circle which can be inscribed around contour L_0 (Fig. 80). Furthermore, we assume $R_0(\beta)$ as the radius vector of contour L_0, where β – vectorial angle; a – radius of circular inscribed around contour L_0; ΔS – region of plane z = 0, which supplements region S_0 to the circle of radius a.

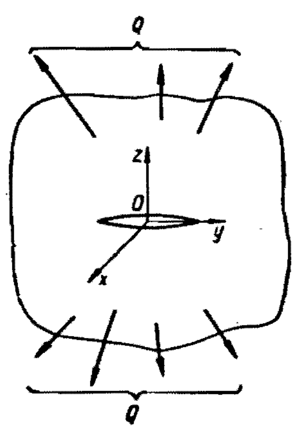

Fig. 79.

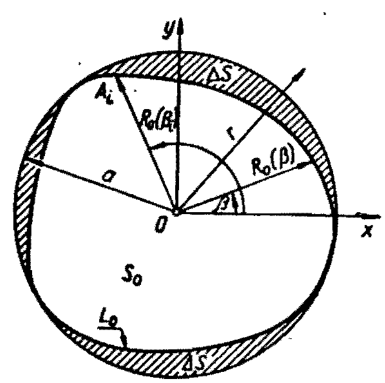

Fig. 80.

Then function

$$\varepsilon(\beta) = a - R_0(\beta) \tag{IX.1}$$

is a nonnegative and periodic function with period 2π.

A plane macrocrack bounded by closed contour L_0 is a crack having a near-circular shape, if the maximum value of function $\varepsilon(\beta)$ is small as compared to radius a.

We assume that the brittle body containing an internal macrocrack which is nearly circular is extended by monotonically increasing external forces Q, directed symmetrically with respect to the planes of the crack. The problem consists in determination of the least value of external stresses $Q = Q_*$ at which the crack L_0 goes into a state of dynamic equilibrium at least in one point of this contour.

The external load $Q_*^{(i)}$, at which the contour of crack L_0 in the neighborhood of point $R_0(\beta_i)$ goes into a state of dynamic equilibrium can be determined from equation (VIII.32).

Thus, resolution of the problem about the limit load for a crack which is nearly circular reduced to determination of elastic stresses $\sigma_z(r, \beta, 0)$ in the neighborhood of the contour of this crack.

According to the results of work [69] a method is developed below for the approximate solution of this class of problems [115, 117, 118].

2. Initial Equation for Determination of Tensile Stresses in the Neighborhood of the Contour of a Crack

Let us determine stresses $\sigma_z(r, \beta, 0)$ in the neighborhood of the contour of a crack S_0 (see Fig. 80). Let us present these stresses in the form

$$\sigma_z(r, \beta, 0) = \begin{cases} \sigma_z^{(0)}(r, \beta, 0) + \sigma_z^{(1)}(r, \beta, 0) & \text{when } r \geqslant a; \\ -g(r, \beta) & \text{when } R_0(\beta) \leqslant r \leqslant a, \end{cases} \quad \text{(IX.2)}$$

where $0 \leqslant \beta \leqslant 2\pi$; $\sigma_z^{(0)}(r, \beta, 0)$ — normal tensile stresses in the plane z = 0, appearing in an unbounded elastic body with internal round crack with radius a when assigned system of loads Q acts on the body; g(r, β) — unknown pressure in region ΔS; $\sigma_z^{(1)}(r, \beta, 0)$ — normal stresses appearing in the plane z = 0 (in plane of location of crack) when r ≥ a as a result of the action of normal pressure g(r, β) on sections $\Delta S[R_0(\beta) \leq r \leq a;\ 0 \leq \beta \leq 2\pi]$ of the surface of the circular crack.

Inasmuch as for our problem plane z = 0 is a plane of symmetry, then calculation of stresses $\sigma_z^{(0)}(r, \beta, 0)$ and $\sigma_z^{(1)}(r, \beta, 0)$ when r ≥ a leads to the solution of the following problem of the theory of elasticity. On surface z = 0 of elastic half-space z ≥ 0 are assigned boundary conditions

$$\begin{aligned} &\tau_{xz}(r, \beta, 0) = \tau_{yz}(r, \beta, 0) = 0 \ \text{where } 0 \leqslant r \leqslant \infty, \\ &w(r, \beta, 0) = 0 \quad \text{where } r \geqslant a, \\ &\sigma_z(r, \beta, 0) = -f(r, \beta) \quad \text{where } r \leqslant a, \end{aligned} \quad \text{(IX.3)}$$

where 0 ≤ β ≤ 2π; w(r, β, 0) — component of shift along axis Oz; f(r, β) — known function.

It is necessary to determine on the basis of boundary conditions (IX.3) stresses $\sigma_z(r, \beta, 0)$ when r ≥ a, 0 ≤ β ≤ 2π.

The solution of such a problem, as is known, is represented by formula (VI.26):

$$\sigma_z(r, \beta, 0) = \frac{1}{\pi^2 \sqrt{r^2 - a^2}} \int_0^{2\pi} \int_0^a \frac{\sqrt{a^2 - \rho^2}\, f(\rho, \alpha)\, \rho\, d\rho\, d\alpha}{r^2 + \rho^2 - 2r\rho\cos(\alpha - \beta)}, \quad \text{(IX.4)}$$

where r ≥ a.

By this formula it is possible to calculate stresses $\sigma_z^{(0)}(r, \beta, 0)$ when r ≥ a:

$$\sigma_z^{(0)}(r, \beta, 0) = \frac{\psi(r, \beta)}{\sqrt{r^2 - a^2}}. \qquad \text{(IX.5)}$$

Here $\psi(r, \beta)$ — is a known regular function,

$$\psi(r, \beta) = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{a} \frac{\sqrt{a^2 - \rho^2}\,[t_1(\rho, \alpha) + q(\rho, \alpha)]\,\rho d\rho d\alpha}{r^2 + \rho^2 - 2r\rho\cos(\alpha - \beta)} + \sqrt{r^2 - a^2}\, q(r, \beta), \qquad \text{(IX.6)}$$

where $t_1(\rho, \alpha)$ is the assigned external pressure applied to the walls of the crack; $q(r, \beta)$ — distribution of normal stresses $\sigma_z(x, y, 0)$, appearing in a solid (without a crack) elastic body in plane $z = 0$ from the action of external forces Q, applied outside the crack. Subsequently function $q(r, \beta)$ will be considered as known. For example, when a body on infinity is extended by constant stresses $\sigma_z(x, y, \infty) = p$, we have

$$t_1(\rho, \alpha) = 0; \quad q(r, \beta) = q_1 = p;$$
$$\psi_1(r, \beta) = \frac{2p}{\pi}\left\{a + \sqrt{r^2 - a^2}\left(\frac{\pi}{2} - \arcsin\frac{a}{r}\right)\right\}. \qquad \text{(IX.7)}$$

If an elastic body is loaded by two equal in magnitude and oppositely directed forces P, applied at points (0, 0, +h) and (0, 0, -h), then

$$q_2(r, \beta) = \frac{P}{4\pi(1-\nu)} \cdot \frac{h}{(r^2 + h^2)^{3/2}}\left(\frac{3h^2}{r^2 + h^2} + 1 - 2\nu\right); \quad t_1(\rho, \alpha) = 0.$$

Hence and on the basis of formula (IX.6) we find

$$\psi_2(r, \beta) = \frac{Pa}{2\pi^2(1-\nu)(r^2 + h^2)^2}\Big[2\{(1-\nu)r^2 + (2-\nu)h^2\} -$$
$$- h^2\frac{r^2 - a^2}{a^2 + h^2} - \frac{h}{a}\{(r^2 + h^2)(1 - 2\nu) + 3h^2\}\sqrt{\frac{r^2 - a^2}{r^2 + h^2}} \times$$
$$\times \operatorname{arctg}\frac{a}{h}\sqrt{\frac{r^2 + h^2}{r^2 - a^2}}\Big] + \sqrt{r^2 - a^2}\, q_2(r, \beta). \qquad \text{(IX.8)}$$

According to formulas (IX.2)-(IX.4) stress $\sigma_z^{(1)}(r, \beta, 0)$ is expressed through unknown pressure $q(r, \beta)$ as:

$$\sigma_z^{(1)}(r, \beta, 0) = \frac{1}{\pi^2\sqrt{r^2 - a^2}} \iint_{(\Delta S)} \frac{\sqrt{a^2 - \rho^2}\, g(\rho, \alpha)\,\rho d\rho d\alpha}{r^2 + \rho^2 - 2r\rho\cos(\alpha - \beta)} \quad (r \geqslant a), \qquad \text{(IX.9)}$$

where ΔS is shown on Fig. 80.

On the basis of formulas (IX.2), (IX.5), and (IX.9) the formula for determination of stress $\sigma_z(r, \beta, 0)$ when $r \geq a$ can be represented in the following form:

$$\sigma_z(r, \beta, 0) = \frac{1}{\sqrt{r^2 - a^2}} \left\{ \psi(r, \beta) + \frac{1}{\pi^2} \iint_{(\Delta S)} \frac{\sqrt{a^2 - \rho^2}\, g(\rho, \alpha)\, \rho d\rho d\alpha}{r^2 + \rho^2 - 2r\rho \cos(\alpha - \beta)} \right\}, \qquad \text{(IX.10)}$$

where

$$r \geqslant a; \quad R_0(\beta) \leqslant \rho \leqslant a; \quad 0 \leqslant \alpha; \quad \beta \leqslant 2\pi.$$

A circle with radius a inscribed around contour L_0 (see Fig. 80) is located in a region of elastic material. Consequently, stresses $\sigma_z(r, \beta, 0)$, determined by formula (IX.10), on contour $r = a$ must be continuous. From this formula it follows that a necessary condition of continuity of stresses $\sigma_z(r, \beta, 0)$ when $r = a$ is the equality

$$\lim_{r \to a} \left\{ \psi(r, \beta) + \frac{1}{\pi^2} \iint_{(\Delta S)} \frac{\sqrt{a^2 - \rho^2}\, g(\rho, \alpha)\, \rho d\rho d\alpha}{r^2 + \rho^2 - 2r\rho \cos(\alpha - \beta)} \right\} = 0. \qquad \text{(IX.11)}$$

Equality (IX.11) constitutes an integral equation for determination of pressure $g(r, \beta)$ or stresses $\sigma_z(r, \beta, 0) = -g(r, \beta)$ in region ΔS.

This approach to determining pressure under the base of a flat stamp located on an elastic half-space and nearly circular in shape was first proposed by M. Ya. Leonov; subsequently it was used in solving certain contact problems of the theory of elasticity [68, 69].

3. Certain Assumptions and Transformations of the Initial Equation

For a crack nearly circular in shape (see Fig. 80), the unknown stress by analogy with formula (IX.5) will be represented in such a form:

$$g(r, \beta) = \frac{\varphi(r, \beta)}{\sqrt{r^2 - R_0^2(\beta)}} \quad (a \geqslant r \geqslant R_0(\beta); \quad 0 \leqslant \beta \leqslant 2\pi), \qquad \text{(IX.12)}$$

where function $\phi(r, \beta)$ is still unknown.

We assume that $\phi(r, \beta)$ is a regular function and, consequently, it can be expanded in a series in powers of $(r - a)$ in the neighborhood of points $r = a$. This is possible if the equation of contour L_0 of the examined crack does not have singular points. Subsequently we assume that over our assumptions with respect to function $\phi(r, \beta)$ are fulfilled. In such case equation (IX.11) can be written as:

$$\psi(a, \beta) = -\frac{1}{\pi^2} \lim_{r \to a} \int_0^{2\pi} \int_{R_0(\alpha)}^{a} \sqrt{\frac{a^2 - \rho^2}{\rho^2 - R_0^2(\alpha)}} \cdot \frac{\varphi(\rho, \alpha)\, \rho d\rho d\alpha}{r^2 + \rho^2 - 2r\rho \cos(\alpha - \beta)}, \tag{IX.13}$$

where $r \geq a$.

We will transform equation (IX.13) just as is done in work [69]. To do this we expand function $\phi(r, \beta)$ in a series in powers of $(r - a)$ in the neighborhood of points $r = a$. Limiting ourselves in this expansion to only the first two components, we obtain

$$\varphi(r, \beta) \approx \varphi(a, \beta) + (r - a)\, \varphi_r'(a, \beta). \tag{IX.14}$$

Placing this expression in equation (IX.13), we find

$$\psi(a, \beta) = -\frac{1}{\pi^2} \lim_{r \to a} \int_0^{2\pi} \int_{R_0(\alpha)}^{a} \sqrt{\frac{a^2 - \rho^2}{\rho^2 - R_0^2(\alpha)}} \times$$
$$\times \frac{[\varphi(a, \alpha) + (\rho - a)\, \varphi_r'(a, \alpha)]\, \rho d\rho d\alpha}{r^2 + \rho^2 - 2r\rho \cos(\alpha - \beta)}. \tag{IX.15}$$

Since $a - \rho \leq \varepsilon_*$, where $\varepsilon_* = \max \varepsilon(\beta)$, we can conclude that equation (IX.15) differs from equation (IX.13) by small quantities — of the order of $\left(\frac{\varepsilon_*}{a}\right)^2$. Furthermore, let us note that to the same accuracy we can write

$$\sqrt{\frac{a^2 - \rho^2}{\rho^2 - R_0^2(\beta)}} = \left[1 + \frac{\varepsilon(\beta)}{4a}\right] \sqrt{\frac{a - \rho}{\rho - R_0(\beta)}}.$$

Consequently, equation (IX.13) for determination of functions $\phi(a, \beta)$ and $\phi'_r(a, \beta)$ can be presented accurate to $\frac{\varepsilon_*}{a}$ inclusively in the form:

$$\psi(a, \beta) = -\frac{1}{\pi^2}\lim_{r\to a}\int_0^{2\pi}\mu(\alpha)\{\varphi(a, \alpha)I_0(r, \alpha, \lambda) + \varphi_r'(a, \alpha)\times \\ \times I_1(r, \alpha, \lambda)\}\,d\alpha, \qquad \text{(IX.16)}$$

where

$$\mu(\alpha) = 1 + \frac{\varepsilon(\alpha)}{4a}; \quad \lambda = \frac{\alpha-\beta}{2}; \qquad \text{(IX.17)}$$

$$\left.\begin{aligned} I_0(r, \alpha, \lambda) &= \int_{R_0(\alpha)}^{a}\sqrt{\frac{a-\rho}{\rho-R_0(\alpha)}}\,\frac{\rho d\rho}{r^2+\rho^2-2r\rho\cos(\alpha-\beta)}; \\ I_1(r, \alpha, \lambda) &= \int_{R_0(\alpha)}^{a}\sqrt{\frac{a-\rho}{\rho-R_0(\alpha)}}\,\frac{\rho(\rho-a)\,d\rho}{r^2+\rho^2-2r\rho\cos(\alpha-\beta)}. \end{aligned}\right\} \qquad \text{(IX.18)}$$

If in expressions (IX.18) we introduce integration variable

$$t^2 = \frac{a-\rho}{\rho-R_0(\alpha)},$$

then after simple transformations we obtain

$$\left.\begin{aligned} I_0(r, \alpha, \lambda) &= -[R_0(\alpha)-a]\int_{-\infty}^{+\infty}\frac{t^2[a+t^2R_0(\alpha)]\,dt}{(1+t^2)(At^4+Bt^2+C)}; \\ I_1(r, \alpha, \lambda) &= -[R_0(\alpha)-a]^2\int_{-\infty}^{+\infty}\frac{t^4[a+t^2R_0(\alpha)]\,dt}{(1+t^2)^2(At^4+Bt^2+C)}, \end{aligned}\right\} \qquad \text{(IX.19)}$$

where

$$\left.\begin{aligned} A &= r^2 + R_0^2(\alpha) - 2rR_0(\alpha)\cos 2\lambda; \\ B &= 2R_0(\alpha)(a - r\cos 2\lambda) + 2r(r - a\cos 2\lambda); \\ C &= C(r, \lambda) = r^2 + a^2 - 2ra\cos 2\lambda. \end{aligned}\right\} \qquad \text{(IX.20)}$$

Integrals (IX.19) can be calculated on the basis of the theory of risidues [156]. Then

$$I_0(r,\alpha,\lambda) = -\pi\left\{1 - \frac{\varepsilon(\alpha) R_0(\alpha) + \chi(r,\lambda) + \sqrt{C(r,\lambda)\chi(r,\lambda)}}{\sqrt{2\chi(r,\lambda)}\sqrt{\sqrt{C(r,\lambda)\chi(r,\lambda)} + R_0(\alpha) D_1(\lambda) + rD_2(\lambda)}}\right\}, \quad \text{(IX.21)}$$

$$I_1(r,\alpha,\lambda) = \pi\left\{\frac{\varepsilon(\alpha) + 2(a - 2r\cos 2\lambda)}{2} + \frac{2rD_1(\lambda)[2R_0(\alpha)\cos 2\lambda - r] - R_0(\alpha)(a^2 - r^2) + (2r\cos 2\lambda - a)\sqrt{C(r,\lambda)\chi(r,\lambda)}}{\sqrt{2\chi(r,\lambda)}\sqrt{\sqrt{C(r,\lambda)\chi(r,\lambda)} + R_0(\alpha) D_1(\lambda) + rD_2(\lambda)}}\right\}, \quad \text{(IX.22)}$$

where parameter $C(r, \lambda)$ is determined by the formula (IX.20);

$$\chi(r,\lambda) = R_0^2(\alpha) + r^2 - 2rR_0(\alpha)\cos 2\lambda;$$
$$D_1(\lambda) = a - r\cos 2\lambda; \quad D_2(\lambda) = r - a\cos 2\lambda. \quad \text{(IX.23)}$$

Taking out in these expressions only terms containing $\varepsilon(\alpha)$ and $\varepsilon^2(\alpha)$, we obtain

$$I_0(r,\alpha,\lambda) \approx \frac{\pi}{2}\left\{\frac{a\varepsilon(\alpha)}{a^2 + r^2 - 2ra\cos 2\lambda} - \frac{3}{4}\,\frac{(r^2 - a^2)\varepsilon^2(\alpha)}{(a^2 + r^2 - 2ra\cos 2\lambda)^2}\right\}, \quad \text{(IX.24)}$$

$$I_1(r,\alpha,\lambda) \approx -\frac{3\pi}{8}\cdot\frac{a\varepsilon^2(\alpha)}{a^2 + r^2 - 2ra\cos 2\lambda}. \quad \text{(IX.25)}$$

When $r = a$ integrals (IX.21) and (IX.22) can be represented in the form:

$$I_0(a,\alpha,\lambda) = -\pi \times \left\{1 - \frac{\sqrt{a}\,[a - R_0(\alpha) + 4R_0(\alpha)\sin^2\lambda + 2\sqrt{\chi(a,\lambda)\sin^2\lambda}]}{2\sqrt{\chi(a,\lambda)}\sqrt{\sqrt{\chi(a,\lambda)\sin^2\lambda} + [R_0(\alpha) + a]\sin^2\lambda}}\right\}; \quad \text{(IX.26)}$$

$$I_1(a,\alpha,\lambda) = -\pi\left\{\frac{R_0(\alpha) - a - 2a(4\sin^2\lambda - 1)}{2} - \frac{a\sqrt{a}\,\{2\sin^2\lambda\,[2R_0(\alpha) - a - 4R_0(\alpha)\sin^2\lambda] + (1 - 4\sin^2\lambda)\sqrt{\chi(a,\lambda)\sin^2\lambda}}{\sqrt{\chi(a,\lambda)}\sqrt{\sqrt{\chi(a,\lambda)\sin^2\lambda} + [R_0(\alpha) + a]\sin^2\lambda}}\right\}, \quad \text{(IX.27)}$$

where

$$\chi(a, \lambda) = [R_0(\alpha) - a]^2 + 4aR_0(\alpha)\sin^2\lambda.$$

We introduce now into consideration the function

$$U(a, \alpha, \lambda) = \frac{1}{\pi^2}\left[1 + \frac{\varepsilon(\alpha)}{4a}\right]\{\varphi(a, \alpha) I_0(a, \alpha, \lambda) + \\ + \varphi'_r(a, \alpha) I_1(a, \alpha, \lambda)\}, \qquad \text{(IX.28)}$$

where values of $I_0(a, \alpha, \lambda)$, $I_1(a, \alpha, \lambda)$ are determined by formulas (IX.26) and (IX.27).

Let us note that function $U(a, \alpha, \lambda)$ is continuous in intervals $0 \le \alpha \le \beta - 2\eta_1$ and $\beta + 2\eta_1 \le \alpha \le 2\pi$, where η_1 is a small quantity. In these intervals functions $I_0(a, \alpha, \lambda)$ and $I_1(a, \alpha, \lambda)$ can be expanded in a series in powers of $\varepsilon(\alpha)$. Disregarding in such an expansion terms whose order of smallness exceeds $\varepsilon(\alpha)$, according to (IX.24) and (IX.25), we obtain

$$I_0(a, \alpha, \lambda) = \frac{\pi\varepsilon(\alpha)}{8a\sin^2\lambda} + 0(\varepsilon_*^2); \quad I_1(a, \alpha, \lambda) = 0(\varepsilon_*^2). \qquad \text{(IX.29)}$$

In the interval $\beta - 2\eta_1 \le \alpha \le \beta + 2\eta_1$ such an expansion of functions $I_0(a, \alpha, \lambda)$ and $I_1(a, \alpha, \lambda)$ is impossible.

Parameter η_1 is determined in this way so that functions $I_0(a, \alpha, \lambda)$ and $I_1(a, \alpha, \lambda)$ when $\alpha = \beta \pm 2\eta_1$ accurate to magnitudes of the order of $\varepsilon(\alpha)$ is satisfied by the following equalities:

$$I_0(a, \beta, \pm\eta_1) = \frac{\pi\varepsilon(\beta)}{8a\sin^2\eta_1}; \quad I_1(a, \beta, \pm\eta_1) = 0. \qquad \text{(IX.30)}$$

On the basis of formulas (IX.26) and (IX.27) one can be certain that equalities (IX.30) are fulfilled to this accuracy [69] if

$$\eta_1 = \frac{1}{2}\sqrt{\frac{\varepsilon(\alpha)}{a}}. \qquad \text{(IX.31)}$$

Using expressions (IX.28), (IX.29) and (IX.31), equation (IX.16) accurate to $\frac{\varepsilon_*}{a}$ inclusively can be written in this form:

$$-\psi(a,\beta)=\frac{1}{8\pi a}\int_0^{\beta-2\eta_1}\frac{\varepsilon(\alpha)\varphi(a,\alpha)}{\sin^2\lambda}d\alpha+\frac{1}{8\pi a}\int_{\beta+2\eta_1}^{2\pi}\frac{\varepsilon(\alpha)\varphi(a,\alpha)}{\sin^2\lambda}d\alpha+$$
$$+\int_{\beta-2\eta_1}^{\beta+2\eta_1}U(a,\alpha,\lambda)\,d\alpha\quad\left[\lambda=\frac{1}{2}(\alpha-\beta)\right]. \tag{IX.32}$$

Inasmuch as η_1 is small, then, not disturbing the accuracy of equation (IX.32), the integral from function $U(a,\alpha,\lambda)$ in this equation can be written as:

$$\int_{\beta-2\eta_1}^{\beta+2\eta_1}U(a,\alpha,\lambda)\,d\alpha=-\frac{4\eta_1}{\pi}[\varphi(a,\beta)+a\varphi_r'(a,\beta)]+$$
$$+\frac{1}{\pi}\left[1+\frac{\varepsilon(\beta)}{4a}\right]\{\varphi(a,\beta)T_0(\eta_1)+a\varphi_r'(a,\beta)T_1(\eta_1)\}. \tag{IX.33}$$

Here

$$T_0(\eta_1)=2\sqrt{a}\int_0^{\eta_1}\frac{\{\varepsilon(\alpha)+4R_0(\alpha)\sin^2\lambda+2\sqrt{\chi(a,\lambda)\sin^2\lambda}\}\,d\lambda}{\sqrt{\chi(a,\lambda)}\sqrt{\sqrt{\chi(a,\lambda)\sin^2\lambda}+[R_0(\alpha)+a]\sin^2\lambda}}; \tag{IX.34}$$

$$T_1(\eta_1)=4\sqrt{a}\int_0^{\eta_1}\frac{\{2\sin^2\lambda\,[2R_0(\alpha)-a-4R_0(\alpha)\sin^2\lambda]+(1-4\sin^2\lambda)\sqrt{\chi(a,\lambda)\sin^2\lambda}\}\,d\lambda}{\sqrt{\chi(a,\lambda)}\sqrt{\sqrt{\chi(a,\lambda)\sin^2\lambda}+[R_0(\alpha)+a]\sin^2\lambda}}, \tag{IX.35}$$

where

$$\chi(a,\lambda)=\varepsilon^2(\alpha)+4aR_0(\alpha)\sin^2\lambda;\ \lambda=\frac{1}{2}(\alpha-\beta).$$

If we consider that functions $\varepsilon(\alpha)$ and $\phi(a,\alpha)$ have a period of 2π with respect to the variable α, then it is easy to show that accurate to ε_* inclusively, the following equality holds:

$$\frac{1}{8\pi a}\left\{\int_0^{\beta-2\eta_1}\frac{\varepsilon(\alpha)\varphi(a,\alpha)}{\sin^2\lambda}d\alpha+\int_{\beta+2\eta_1}^{2\pi}\frac{\varepsilon(\alpha)\varphi(a,\alpha)}{\sin^2\lambda}d\alpha\right\}=$$

$$=\frac{1}{4\pi a}\left\{\int_0^{2\pi}\frac{d}{da}[\varepsilon(\alpha)\varphi(a,\alpha)]\operatorname{ctg}\frac{\alpha-\beta}{2}\,d\alpha-\right.$$

$$\left.-\int_{\beta-2\eta_1}^{\beta+2\eta_1}\frac{d}{da}[\varepsilon(\alpha)\varphi(a,\alpha)]\operatorname{ctg}\frac{\alpha-\beta}{2}\,d\alpha\right\}+\frac{\varepsilon(\beta)\varphi(a,\beta)}{2\pi a}\cdot\frac{1}{\eta_1}. \qquad \text{(IX.36)}$$

Functions $\varepsilon(\alpha)$ and $\phi(a, \alpha)$ are regular, and η_1 is a small quantity which is determined by equality (IX.31), therefore it is possible to show that

$$\int_{\beta-2\eta_1}^{\beta+2\eta_1}\frac{d}{da}[\varepsilon(\alpha)\varphi(a,\alpha)]\operatorname{ctg}\frac{\alpha-\beta}{2}\,d\alpha=0(\varepsilon_*^2). \qquad \text{(IX.37)}$$

Thus, on the basis of formulas (IX.33), (IX.36), and (IX.37) equation (IX.32) accurate to magnitudes of the order of $\frac{\varepsilon_*}{a}$ inclusively can be represented in the following form:

$$\psi(a,\beta)=\frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{da}[\varepsilon(\alpha)\varphi(a,\alpha)]\operatorname{ctg}\frac{\beta-\alpha}{2}\,d\alpha+$$

$$+\frac{4\eta_1}{\pi}[\varphi(a,\beta)+a\varphi_r'(a,\beta)]-\frac{\varepsilon(\beta)\varphi(a,\beta)}{2\pi a}\cdot\frac{1}{\eta_1}-$$

$$-\frac{1}{\pi}\left[1+\frac{\varepsilon(\beta)}{4a}\right]\{\varphi(a,\beta)T_0(\eta_1)+a\varphi_r'(a,\beta)T_1(\eta_1)\}, \qquad \text{(IX.38)}$$

where functions $T_0(\eta_1)$ and $T_1(\eta_1)$ are determined by formulas (IX.34) and (IX.35).

Inasmuch as equation (IX.38) is constructed accurate to small quantities of the order of $\frac{\varepsilon(\beta)}{a}$ inclusively, then it is available to calculate to the same accuracy functionals $T_0(\eta_1)$ and $T_1(\eta_1)$, which are represented by formulas (IX.34) and (IX.35). Having this in mind, we will examine the following integrals:

$$H_0(\lambda)=\int\frac{\{\varepsilon+4(a-\varepsilon)\sin^2\lambda+2\sqrt{\sin^2\lambda\,[\varepsilon^2+4a(a-\varepsilon)\sin^2\lambda]}\}\,d\lambda}{\Theta_0(\lambda)\sqrt{\Theta_0(\lambda)\sqrt{\sin^2\lambda}+(2a-\varepsilon)\sin^2\lambda}}, \qquad \text{(IX.39)}$$

$$H_1(\lambda)=\int\frac{\{2\sin^2\lambda\,[a-2\varepsilon-4(a-\varepsilon)\sin^2\lambda]+(1-4\sin^2\lambda)\sqrt{\sin^2\lambda}\,\Theta_0(\lambda)\}\,d\lambda}{\Theta_0(\lambda)\sqrt{\Theta_0(\lambda)\sqrt{\sin^2\lambda}+(2a-\varepsilon)\sin^2\lambda}}, \qquad \text{(IX.40)}$$

where

$$Э_0(\lambda) = \sqrt{\varepsilon^2 + 4a(a - \varepsilon)\sin^2\lambda}; \ \varepsilon = \varepsilon(\alpha).$$

If integrals (IX.39) and (IX.40) are calculated, the value of functions $T_0(\eta_1)$ and $T_1(\eta_1)$ is found by the formulas

$$T_0(\eta_1) = 2\sqrt{a}\left\{H_0(\eta_1) - \lim_{\lambda \to 0} H_0(\lambda)\right\}; \tag{IX.41}$$

$$T_1(\eta_1) = 4\sqrt{a}\left\{H_1(\eta_1) - \lim_{\lambda \to 0} H_1(\lambda)\right\}. \tag{IX.42}$$

To calculate integrals (IX.39) and (IX.40) we introduce a new integration variable

$$x = \sqrt{\frac{\sqrt{\varepsilon^2 + 4a(a-\varepsilon)\sin^2\lambda} + \varepsilon\cos\lambda}{\sin\lambda}} \quad \text{or} \quad \operatorname{tg}\lambda = \frac{2\varepsilon x^2}{x^4 - (2a-\varepsilon)^2}. \tag{IX.43}$$

Then integral $H_0(\lambda)$ can be presented in the following form

$$H_0^*(x) = -2\sqrt{2}\int \frac{x^8 + 4\varepsilon x^6 - 2(4a^2 - 12a\varepsilon + 7\varepsilon^2)x^4 + 4\varepsilon(2a-\varepsilon)^2 x^2 + (2a-\varepsilon)^4]}{(2a - \varepsilon + x^2)[x^8 - 2(4a^2 - 4a\varepsilon - \varepsilon^2)x^4 + (2a-\varepsilon)^4]}\,dx,$$

$$H_0^*(x) = H_0\left(\operatorname{arctg}\frac{2\varepsilon x^2}{x^4 - (2a-\varepsilon)^2}\right). \tag{IX.44}$$

Expanding the integrand in equality (IX.44) into elementary fractions, after calculation of the integrals from every component, we find

$$H_0^*(x) = 2\sqrt{2}\left\{\frac{1}{\sqrt{2a-\varepsilon}}\operatorname{arctg}\frac{x}{\sqrt{2a-\varepsilon}} - \right.$$

$$\left. - \frac{1}{\sqrt{2a}}\operatorname{arctg}\frac{\sqrt{2}(\sqrt{a} + \sqrt{a-\varepsilon})x}{(2a-\varepsilon) - x^2} - \frac{1}{\sqrt{2a}}\operatorname{arctg}\frac{\sqrt{2}(\sqrt{a} - \sqrt{a-\varepsilon})x}{(2a-\varepsilon) - x^2}\right\}.$$

Passing in this expression to variable λ in accordance with formula (IX.43), we obtain

$$H_0(\lambda) = 2\sqrt{2}\left\{\frac{1}{\sqrt{2a-\varepsilon}} \operatorname{arctg} \sqrt{\frac{\vartheta_0(\lambda)+\varepsilon\cos\lambda}{(2a-\varepsilon)\sin\lambda}} - \right.$$

$$- \frac{1}{\sqrt{2a}} \operatorname{arctg} \frac{\sqrt{2[\vartheta_0(\lambda)+\varepsilon\cos\lambda]}\,(\sqrt{a}+\sqrt{a-\varepsilon})^2\sin\lambda}{(2a-\varepsilon)\sin\lambda - \varepsilon\cos\lambda - \vartheta_0(\lambda)} -$$

$$\left. - \frac{1}{\sqrt{2a}} \operatorname{arctg} \frac{\sqrt{2[\vartheta_0(\lambda)+\varepsilon\cos\lambda]}\,(\sqrt{a}-\sqrt{a-\varepsilon})^2\sin\lambda}{(2a-\varepsilon)\sin\lambda - \varepsilon\cos\lambda - \vartheta_0(\lambda)}\right\}. \qquad \text{(IX.45)}$$

Just as before, substituting the integration variable in expression (IX.40) according to formula (IX.43) we find

$$H_1^*(x) = -4\sqrt{2}\,\varepsilon \int \left\{ \frac{[x^8 - 2(4a^2-4a\varepsilon-\varepsilon^2)x^4 + (2a-\varepsilon)^4] \times [x^4 + 4(a-2\varepsilon)x^2 + (2a-\varepsilon)^2]}{[(2a-\varepsilon)+x^2][x^8 - 2(4a^2-4a\varepsilon-\varepsilon^2)x^4 + (2a-\varepsilon)^4]^2} - \right.$$

$$\left. - \frac{16\varepsilon^2[x^4 + 4(a-\varepsilon)x^2 + (2a-\varepsilon)^2]x^4}{[(2a-\varepsilon)+x^2][x^8 - 2(4a^2-4a\varepsilon-\varepsilon^2)x^4 + (2a-\varepsilon)^4]^2} \right\} x^2 dx.$$

In this expression expanding the integrand into prime factors, we obtain the following expression:

$$H_1^*(x) = -2\sqrt{2}\int \frac{dx}{x^2+(2a-\varepsilon)} + \frac{2a\sqrt{2a(a-\varepsilon)}+(2a^2-\varepsilon^2)}{2a\sqrt{2a(a-\varepsilon)}} \times$$

$$\times \left\{\int \frac{dx}{x^2+mx+(2a-\varepsilon)} + \int \frac{dx}{x^2-mx+(2a-\varepsilon)}\right\} +$$

$$+ \frac{2a\sqrt{a(a-\varepsilon)}-(2a^2-\varepsilon^2)}{2a\sqrt{2a(a-\varepsilon)}} \left\{\int \frac{dx}{x^2+nx+(2a-\varepsilon)} + \int \frac{dx}{x^2-nx+(2a-\varepsilon)}\right\} +$$

$$+ \frac{\varepsilon^2(a+\sqrt{a(a-\varepsilon)})}{ma\sqrt{2a(a-\varepsilon)}} \left\{\int \frac{xdx}{[x^2+mx+(2a-\varepsilon)]^2} - \right.$$

$$\left. - \int \frac{xdx}{[x^2-mx+(2a-\varepsilon)]^2}\right\} - \frac{\varepsilon^2(a-\sqrt{a(a-\varepsilon)})}{na\sqrt{2a(a-\varepsilon)}} \times$$

$$\times \left\{\int \frac{xdx}{[x^2+nx+(2a-\varepsilon)]^2} - \int \frac{xdx}{[x^2-nx+(2a-\varepsilon)]^2}\right\}, \qquad \text{(IX.46)}$$

where

$$m = \sqrt{2(2a-\varepsilon) - 4\sqrt{a(a-\varepsilon)}};\ n = \sqrt{2(2a-\varepsilon) + 4\sqrt{a(a-\varepsilon)}}.$$

Calculating the integrals in expression (IX.46), and carrying out simple transformations, we find

$$H_1^*(x) = -\frac{2\sqrt{2}}{\sqrt{2a-\varepsilon}} \operatorname{arctg} \frac{x}{\sqrt{2a-\varepsilon}} + $$
$$+\frac{1}{a\sqrt{2a(a-\varepsilon)}}\left[\frac{2a\sqrt{a(a-\varepsilon)} + (2a^2-\varepsilon^2)}{n} - \frac{2\varepsilon^2(a+\sqrt{a(a-\varepsilon)})}{n^3}\right] \times$$
$$\times \operatorname{arctg} \frac{nx}{(2a-\varepsilon)-x^2} + \frac{1}{a\sqrt{2a(a-\varepsilon)}}\left[\frac{2a\sqrt{a(a-\varepsilon)} - (2a^2-\varepsilon^2)}{m} + \right.$$
$$\left. + \frac{2\varepsilon^2(a-\sqrt{a(a-\varepsilon)})}{m^3}\right] \operatorname{arctg} \frac{mx}{(2a-\varepsilon)-x^2} -$$
$$- \frac{\varepsilon\sqrt{2}}{a} \cdot \frac{x[(2a-\varepsilon)-x^2][x^4+4ax^2+(2a-\varepsilon)^2]}{x^8-2(4a^2-4a\varepsilon-\varepsilon^2)x^4+(2a-\varepsilon)^4}, \qquad \text{(IX.47)}$$

where values of x are determined by equality (IX.43); $\varepsilon = \varepsilon(\alpha)$.

With relationships (IX.41), (IX.42), (IX.45) and (IX.47) evident expressions of functions $T_0(\eta_1)$ and $T_1(\eta_1)$ from parameter η_1 can be obtained. Expanding then functions $T_0(\eta_1)$ and $T_1(\eta_1)$ in a series in powers of η_1 and keeping only terms with η_1 and η_1^2, we obtain

$$\left.\begin{aligned} T_0(\eta_1) &= \pi(1-\eta_1^2) + 2\eta_1 + 0(\eta_1^3); \\ T_1(\eta_1) &= 4\eta_1 - 2\pi\eta_1^2 + 0(\eta_1^3). \end{aligned}\right\} \qquad \text{(IX.48)}$$

Subsequently expressions (IX.48) will be used for final transformation of equation (IX.38).

4. Integral Equation of the Problem and the Method for Solving it

On the basis of formulas (IX.31) and (IX.48) equation (IX.38) after simple transformations can be presented accurate to $\frac{\varepsilon_*}{a}$ in the following form:

$$\psi(a, \beta) = -\varphi(a, \beta) + \frac{1}{2}\varepsilon(\beta)\varphi_r'(a, \beta) +$$
$$+ \frac{1}{4\pi a}\int_0^{2\pi} \frac{d}{d\alpha}[\varepsilon(\alpha)\varphi(a, \alpha)] \operatorname{ctg}\frac{\beta-\alpha}{2}\, d\alpha. \qquad \text{(IX.49)}$$

Equation (IX.49) is an integral differential singular equation with respect to function $\phi(a, \beta)$. This is also the desired approximate equation of the problem about determination of stresses $\sigma_z(r, \beta, 0)$

in the neighborhood of the contour of a crack; it is analogous to the equation obtained in work [69] for determination of contact stresses under the base of a flat stamp which is nearly circular in shape.

During the derivation of equation (IX.49) we disregarded quantities whose order of smallness is higher than $\frac{\varepsilon_*(\beta)}{a}$. Therefore the solution of this equation should also be sought with the same degree of accuracy. This solution is easy to construct using the method of successive approximations. For this we write equation (IX.49) in the form

$$\varphi(a, \beta) = -\psi(a, \beta) + \frac{1}{2}\varepsilon(\beta)\varphi_r'(a, \beta) + \\ + \frac{1}{4\pi a}\int_0^{2\pi} \frac{d}{d\alpha}[\varepsilon(\alpha)\varphi(a, \alpha)]\operatorname{ctg}\frac{\beta-\alpha}{2}\,d\alpha. \qquad \text{(IX.50}$$

By conditions of the problem it is assumed that functions $\varepsilon(\beta)$ and $\phi(r, \beta)$ are regular in interval $[0 \leq \beta \leq 2\pi]$. Therefore accurate to quantities of the order of $\varepsilon(\beta)$ from equation (IX.50) we obtain

$$\varphi_0(a, \beta) = -\psi(a, \beta). \qquad \text{(IX.51)}$$

Inasmuch as radius a of the circle inscribed around contour L_0 of a crack is selected arbitrarily, in particular, it can change in certain small interval $(a + \Delta a)$, where $\Delta a \geq 0$, then for all values of r close to $r = a$, accurate to small quantities of the order of $\varepsilon(\beta)$ the following equality will hold:

$$\varphi_0(r, \beta) = -\psi(r, \beta); \quad \varphi_r'(a, \beta) = -\psi_r'(a, \beta). \qquad \text{(IX.52)}$$

Taking equalities (IX.51) and (IX.52) as the zero approximations of the functions respectively $\phi(a, \beta)$ and $\phi_r'(a, \beta)$ and putting these expressions in the right side of equation (IX.50), we obtain

$$\varphi(a, \beta) = -\psi(a, \beta) - \frac{1}{2}\varepsilon(\beta)\psi_r'(a, \beta) - \\ - \frac{1}{4\pi a}\int_0^{2\pi} \frac{d}{d\alpha}[\varepsilon(\alpha)\psi(a, \alpha)]\operatorname{ctg}\frac{\beta-\alpha}{2}\,d\alpha. \qquad \text{(IX.53)}$$

This formula determines function $\phi(a, \beta)$ accurate to small quantities of the order of $\frac{\varepsilon_*}{a}$ inclusively, i.e., gives a solution to equation (IX.49) with the necessary accuracy.

5. Basic Formulas for Determination of Rupture Stresses and Limit Load

In accordance with formulas (IX.2) and (IX.12) normal stresses $\sigma_z(r, \beta, 0)$ in region ΔS (see Fig. 80) are expressed by the equality

$$\sigma_z(r, \beta, 0) = -g(r, \beta) = \frac{-\varphi(r, \beta)}{\sqrt{r^2 - R_0^2(\beta)}}, \qquad \text{(IX.54)}$$

where $R_0(\beta)$ – the radius vector of the contour of the examined crack in polar system of coordinates, $0 \leq \beta \leq 2\pi$, $R_0(\beta) \leq r \leq a$.

On the basis of formulas (IX.14) and (IX.53) function $\phi(r, \beta)$ approximately (accurate to $\frac{\varepsilon_*}{a}$ inclusively) can be presented as:

$$-\varphi(r, \beta) = \psi(a, \beta) + \frac{1}{2}\varepsilon(\beta)\psi_r'(a, \beta) + (r - a)\psi_r'(a, \beta) + \\ + \frac{1}{4\pi a}\int_0^{2\pi} \frac{d}{d\alpha}[\varepsilon(\alpha)\psi(a, \alpha)]\operatorname{ctg}\frac{\beta - \alpha}{2}\,d\alpha, \qquad \text{(IX.55)}$$

where function $\psi(r, \beta)$ is determined by the formula (IX.6).

Thus, on the basis of equalities (IX.54) and (IX.55) accurate to $\frac{\varepsilon_*}{a}$ inclusively we will obtain the following formula for determination of stresses of region ΔS:

$$\sigma_z(r, \beta, 0) = \frac{1}{\sqrt{r^2 - R_0^2(\beta)}}\Big\{\psi(a, \beta) + \frac{1}{2}\varepsilon(\beta)\psi_r'(a, \beta) + \\ + (r - a)\psi_r'(a, \beta) + \frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{d\alpha}[\varepsilon(\alpha)\psi(a, \alpha)]\operatorname{ctg}\frac{\beta - \alpha}{2}\,d\alpha\Big\}, \qquad \text{(IX.56)}$$

where $R_0(\beta) \leq r \leq a$; $0 \leq \beta \leq 2\pi$; function $\psi(r, \beta)$ is determined by the formula (IX.6).

Consequently, for a preassigned external load Q, applied to a brittle body having a nearly circular macrocrack (see Figs. 79 and 80), when $R_0(\beta) \leq r \leq a$ stresses $\sigma_z(r, \beta, 0)$ are determined by the formula (IX.56), and when $r > a$ by formulas (IX.2), (IX.5) and (IX.9).

Let us turn to the composition of conditions for determination of limit load for a crack having a nearly circular crack. To do this we will examine certain point A_i on the contour of the crack (Fig. 81), where $R_0(\beta_i)$ is the radius vector of this point.

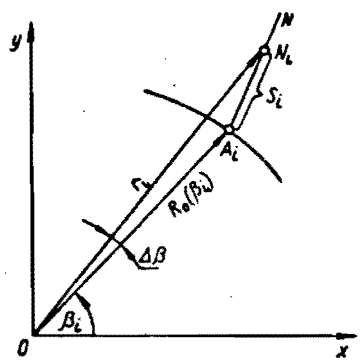

Fig. 81.

If at point A_i we drew normal A_iN to the contour of the crack, then distance $s_i = \overline{A_iN_i}$ between point A_i and point of normal N_i is expressed by the equality:

$$s_i^2 = s_i(\Delta\beta) = r_i^2 + R_0^2(\beta_i) - 2R_0(\beta_i)\, r_i \cos\Delta\beta. \qquad \text{(IX.57)}$$

Here

$$r_i = r(\beta_i + \Delta\beta) = \frac{R_0(\beta_i)\, R_0'(\beta_i)}{x_0'(\beta_i)\cos(\beta_i + \Delta\beta) + y_0'(\beta_i)\sin(\beta_i + \Delta\beta)}, \qquad \text{(IX.58)}$$

where

$$x_0(\beta_i) = R_0(\beta_i)\cos\beta_i; \quad y_0(\beta_i) = R_0(\beta_i)\sin\beta_i. \qquad \text{(IX.58a)}$$

Stresses at point N_i, according to expression (IX.56), are expressed by the formula

$$\sigma_z(r_i,\ \beta_i+\Delta\beta,\ 0)=\frac{1}{\sqrt{r_i^2-R_0^2(\beta_i+\Delta\beta)}}\Big\{\psi(a,\ \beta_i+\Delta\beta)+$$
$$+\frac{1}{2}\varepsilon(\beta_i+\Delta\beta)\psi_r'(a,\ \beta_i+\Delta\beta)+(r_i-a)\psi_r'(a,\ \beta_i+\Delta\beta)+$$
$$\ldots+\frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{d\alpha}[\varepsilon(\alpha)\psi(a,\ \alpha)]\operatorname{ctg}\frac{\beta_i+\Delta\beta-\alpha}{2}\,d\alpha\Big\}. \qquad \text{(IX.59)}$$

Putting expressions (IX.57)-(IX.59) in equation (VII.32) and noticing (see Fig. 81) that the tendency of function s_i to zero is equivalent to $\Delta\beta_i \to 0$, we obtain

$$\lim_{\Delta\beta\to 0}\frac{\sqrt{s_i(\Delta\beta)}\,\pi}{\sqrt{r_i^2-R_0^2(\beta_i+\Delta\beta)}}\Big\{\psi_*(a,\ \beta_i+\Delta\beta)+\frac{1}{2}\varepsilon(\beta_i+\Delta\beta)\times$$
$$\times\psi_{*,r}'(a,\ \beta_i+\Delta\beta)+(r_i-a)\psi_{*,r}'(a,\ \beta_i+\Delta\beta)+\frac{1}{4\pi a}\times$$
$$\times\int_0^{2\pi}\frac{d}{d\alpha}[\varepsilon(\alpha)\psi_*(a,\ \alpha)]\operatorname{ctg}\frac{\beta_i+\Delta\beta-\alpha}{2}\,d\alpha\Big\}=K.$$

Carrying out in this equality passage to the limit when $\Delta\beta \to 0$ and taking into account equality $R_0(\beta_i) - a = -\varepsilon(\beta_i)$, we finally find

$$\frac{\pi}{\sqrt{2R_0(\beta_i)}}\Big\{\psi_*(a,\ \beta_i)-\frac{1}{2}\varepsilon(\beta_i)\psi_{*,r}'(a,\ \beta_i)+$$
$$+\frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{d\alpha}\ \varepsilon(\alpha)\psi_*(a,\ \alpha)]\operatorname{ctg}\frac{\beta_i-\alpha}{2}\,d\alpha\Big\}=K, \qquad \text{(IX.60)}$$

where $\psi_*(a, \beta)$ designates function $\psi(a, \beta)$, represented by formula (IX.6), for the external $Q = Q_*^{(i)}$.

Equation (IX.60) is the fundamental equation for approximate determination (accurate to $\frac{\varepsilon_*}{a}$ inclusively) of the limit load for point $R_0(\beta_i)$ of the contour of a plane isolated, nearly circular crack (Fig. 82).

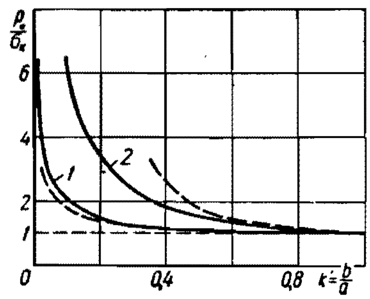

Fig. 82.

6. The Case of a Plane Elliptical Crack

Let us examine the example when in a brittle isotropic body there is an elliptic isolated crack and the body at infinity is extended by stresses $\sigma_z(r, \beta, \infty) = p$ (see Fig. 76), directed perpendicularly to the plane of the crack. Let us determine the limit stresses $p = p_*$ for points of the minor and major axes of the ellipse. Furthermore, we will compare values of p_*, calculated on the basis of the method of approximation developed in this chapter, with values of p_* obtained on the basis of exact solution of the problem [see formula (VIII.44) and (VIII.45)].

For our example the equation of the contour of the crack (see Fig. 76) in a polar system of coordinates will be written as:

$$R_{\text{э}}(\beta) = \frac{b}{\sqrt{1 - k^2 \cos^2 \beta}}; \quad k^2 = \frac{a^2 - b^2}{a^2}, \qquad \text{(IX.61)}$$

where a and b are the major and minor semiaxis of the ellipse.

In accordance with formula (IX.1) we have

$$\varepsilon_{\text{э}}(\beta) = a - \frac{b}{\sqrt{1 - k^2 \cos^2 \beta}}, \qquad \text{(IX.62)}$$

where a is the radius of the circle inscribed around the contour of the crack.

Placing expression (IX.62) in equality (IX.60), we obtain

$$\frac{\pi}{\sqrt{2R_9(\beta_i)}}\Big\{\psi_*(a,\ \beta_i)-\frac{1}{2}\varepsilon_3(\beta_i)\psi'_{*r}(a,\ \beta_i)+$$

$$+\frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{d\alpha}[\varepsilon_3(\alpha)\psi_*(a,\ \alpha)]\operatorname{ctg}\frac{\beta_i-\alpha}{2}d\alpha\Big\}=K. \quad \text{(IX.63)}$$

For the external load $\sigma_z(r,\ \beta,\ \infty) = p$ function $\psi(r,\ \beta)$ is determined by formula (IX.7), using which we find

$$\psi_1(a,\ \beta)=\frac{2ap}{\pi};\ \psi'_{1r}(a,\ \beta)=\frac{4p}{\pi}. \quad \text{(IX.63a)}$$

Putting these values in equality (IX.63) and making certain transformations, we obtain

$$\frac{2p_*^{(\beta)}}{\sqrt{2R_9(\beta)}}\Big\{R_9(\beta)+\frac{bk^2}{4\pi}\int_0^{2\pi}\frac{\cos\alpha\sin\alpha}{\sqrt{(1-k^2\cos^2\alpha)^3}}\operatorname{ctg}\frac{\beta-\alpha}{2}d\alpha\Big\}=K.$$

Hence

$$p_*^{(\beta)}=\frac{K\sqrt{2R_9(\beta)}}{2[R_9(\beta)+I(\beta)]}, \quad \text{(IX.64)}$$

where

$$I(\beta)=\frac{bk^2}{4\pi}\int_0^{2\pi}\frac{\cos\alpha\sin\alpha}{\sqrt{1-k^2\cos^2\alpha}}\operatorname{ctg}\frac{\beta-\alpha}{2}d\alpha. \quad \text{(IX.65)}$$

Integral (IX.65) can be expressed through full elliptic integrals

$$I(\beta)=\frac{b}{\pi}\Big\{K(k)-\frac{E(k)}{1-k^2\cos^2\beta}-\frac{k^2\sin^2\beta}{1-k^2\cos^2\beta}\Pi(n,\ k)\Big\}, \quad \text{(IX.65a)}$$

when $n = -\frac{1}{\cos^2\beta}$, $K(k)$, $E(k)$, and $\Pi(n,\ k)$ are the usual designations of full elliptic integrals of the first, second and third kind respectively.

Here

$$K(k)=\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}};\quad E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\alpha}\,d\alpha;$$

$$\Pi(n,\ k)=\int_0^{\pi/2}\frac{d\alpha}{(1+n\sin^2\alpha)\sqrt{1-k^2\sin^2\alpha}};\quad k^2=\frac{a^2-b^2}{a^2},$$

where

$$n = -\frac{1}{\cos^2 \beta}.$$

Inasmuch as resolution of the problem consists in finding of values of the limit load for points of the minor and major axes of the ellipse, it is necessary to calculate the value of the integral $I(\beta)$ when $\beta = 0$ and $\beta = \frac{\pi}{2}$. In this case on the basis of formula (IX.65) we will have

$$I(0) = -\frac{bk^2}{4\pi(1-k^2)} \int_0^{2\pi} \frac{\sin^2 \alpha}{\sqrt{1-k^2\cos^2\alpha}}\, d\alpha = = -\frac{b}{\pi(1-k^2)}\left[E(k) - \frac{b^2}{a^2} K(k)\right]; \quad \text{(IX.66)}$$

$$I\left(\frac{\pi}{2}\right) = \frac{bk^2}{4\pi} \int_0^{2\pi} \frac{\cos^2 \alpha\, d\alpha}{\sqrt{1-k^2\cos^2\alpha}} = = \frac{b}{\pi}[K(k) - E(k)].$$

According to expressions (IX.64), (IX.65a), and (IX.66) we obtain

$$p_*^{(0)} \approx \frac{\pi k_1 \sigma_k}{\pi k_1 + k_1^2 K(k) - E(k)}; \quad \text{(IX.67)}$$

$$p_*^{\left(\frac{\pi}{2}\right)} \approx \frac{\pi \sigma_k}{\sqrt{k_1}\,[\pi + K(k) - E(k)]}, \quad \text{(IX.68)}$$

where

$$k' = k_1 = \frac{b}{a}; \quad \sigma_k = \frac{K}{\sqrt{2a}} = \sqrt{\frac{\pi E \gamma}{2(1-\nu^2)a}}.$$

Values of $p_*^{(0)}$ and $p_*^{(\pi/2)}$, calculated on the basis of exact solution of this problem (see Chap. VIII), are expressed by the formulas

$$p_*^{(0)} = \frac{2\sigma_k}{\pi k_1} E(k); \quad p_*^{\left(\frac{\pi}{2}\right)} = \frac{2\sigma_k}{\pi\sqrt{k_1}} E(k). \quad \text{(IX.69)}$$

To compare the solutions of (IX.67)-(IX.69) on Fig. 82 the change of values of $\frac{p_*}{\sigma_k}$ as a function of relationships between semiaxes (b/a) of an elliptic crack is graphed, where solid lines (1 and 2) correspond to formulas (IX.69), and the dotted to formulas (IX.67) and (IX.68). Values of $\frac{p_*}{\sigma_k}$ obtained on the basis of approximate formulas (IX.67) and (IX.68) will agree well with values of $\frac{p_*}{\sigma_k}$ calculated by formulas (IX.69) for all values of $\frac{b}{a} > 0.5$.

Thus, the examined example shows that the above method can be used for approximate determination of the limit load for cases when the region occupied by the crack is not very close to circular, i.e., ε_* is commensurable with radius a of a circumscribed circle.

Let us turn to consideration of the case when a brittle body containing an internal elliptic crack is extended by two concentrated forces P (Fig. 83), applied (symmetrically with respect to planes are rubbed) in points (0, 0, +h), (0, 0, -h). Let us determine the force $P = P_*$ after which, points lying on the minor axis of the contour of the crack go into a state of dynamic equilibrium.

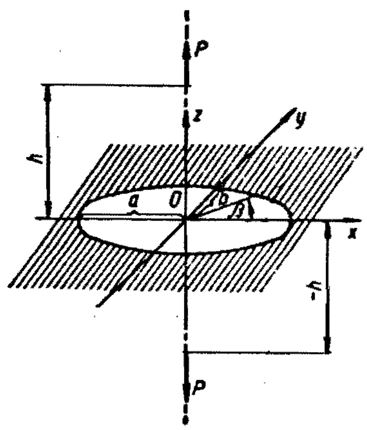

Fig. 83.

For this problem function $\psi(r, \beta)$ is determined by expression (IX.8) on the basis of which we have

$$\psi_2(a,\ \beta) = \frac{P}{\pi^2(1-\nu)a}\left\{\frac{1-\nu}{1+n^2} - \frac{n^2}{(1+n^2)^2}\right\};$$
$$\psi_2'(a,\ \beta) = \frac{P\,[2(2-\nu)n^4 - 6n^2 - 2(1-\nu)]}{\pi^2 a^2(1-\nu)(1+n^2)^3}, \qquad \text{(IX.70)}$$

where $n^2 = \frac{h^2}{a^2}$.

Using further formulas (IX.62) and (IX.63), for determination of the limit load $P = P_*$ we obtain

$$P_*^{(\beta)} = \frac{K\sqrt{2R_9(\beta)}}{\Lambda(\beta)}. \qquad \text{(IX.71)}$$

Here

$$\Lambda(\beta) = \pi\left\{\psi_0 - \frac{1}{2}[a - R_9(\beta)]\,\psi_0' + \frac{\psi_0}{a}\,I(\beta)\right\},$$
$$P\psi_0 = \psi_2(a,\ \beta);\quad P\psi_0' = \psi_2'(a,\ \beta);$$

$I(\beta)$ is determined by the formula (IX.65).

For points located on the minor axis of an elliptic crack have $\beta = \frac{\pi}{2}$. In this case integral $I(\beta)$ is calculated by the formula (IX.66). Consequently, on the basis of dependences (IX.61), (IX.66), and (IX.70), function $\Lambda(\beta)$ when $\beta = \frac{\pi}{2}$ can be presented in the form:

$$\Lambda\left(\frac{\pi}{2}\right) = \pi\left\{\psi_0 + \frac{k_1}{\pi}\,\psi_0[K(k) - E(k)] - \frac{a}{2}(1-k_1)\,\psi_0'\right\}.$$

Placing this expression in equality (IX.71), we find

$$\frac{P_*^{\pi/2}}{P_k} = G(\nu,\ n,\ k_1) = \frac{(1-\nu)\sqrt{k_1}}{A(\nu,\ n)\left\{1 + \frac{k_1}{\pi}[K(k) - E(k)]\right\} - (1-k_1)\,B(\nu,\ n)}, \qquad \text{(IX.72)}$$

where

$$P_k = \pi K\sqrt{2a^3};\quad k_1 = \frac{b}{a};\quad n = \frac{h}{a};\quad k^2 = 1 - k_1^2;$$
$$A(\nu,\ n) = \frac{(1-\nu)(1+n^2) + n^2}{(1+n^2)^2};\quad K = \sqrt{\frac{\pi E\gamma}{1-\nu^2}};$$
$$B(\nu,\ n) = \frac{(2-\nu)n^4 - 3n^2 - (1-\nu)}{(1+n^2)^3};$$

$K(k)$ and $E(k)$ are fully elliptic integrals of the first and second kind respectively.

On Fig. 84 the change of function $G(\nu, n, k_1)$ when $\nu = 0.3$ for certain values of parameter k_1 depending upon the ratio $n = \frac{h}{a}$, where $k_1 = k'$.

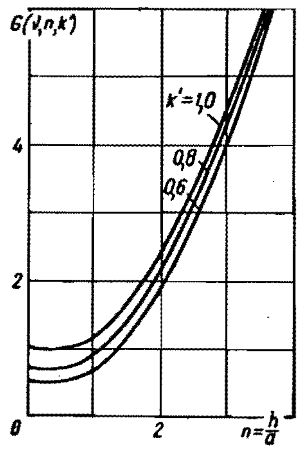

Fig. 84.

When k_1, i.e., if an unbounded brittle body is weakened by a plane circular crack of radius a (see Fig. 76) when a = b from formula (IX.72) we find

$$P_* = \frac{\pi(1-\nu) K \sqrt{2a^3}(1+n^2)^2}{(1-\nu)(1+n^2)+n^2}. \quad \text{(IX.73)}$$

This formula was composed by other means in work [5].

Let us examine an unbounded brittle body weakened by an internal isolated circular crack (plane z = 0, Fig. 85) of radius r_0 with center at point 0_1. Let us assume that such a body is extended by two equal concentrated forces P whose line of action passes through certain point 0, a distance $\delta_1 = 0_1 0$ from the center of the crack. Let us introduce system of cartesian coordinates Oxyz; the origin of the coordinates coincides with point 0, and the x-axis includes segment $0_1 0$. Let us assume that forces P are applied to the body in points with coordinates (0, 0, h) and (0, 0, -h).

We determine the maximum (critical) value of forces $P = P_*$.

For approximate solution of this problem we proceed in the following way. Let us draw in plane xOy a circle with radius $a = r_0 + \delta_1$, its center at point 0 and will consider the contour L_0 of the crack as near to a circle with radius a. In a polar system of coordinates with center at point 0 the equation of contour L_0 of the examined crack can be written as:

$$R_0(\beta) = R(\beta) = \sqrt{r_0^2 - \delta_1^2 \sin^2 \beta} - \delta_1 \cos \beta, \qquad \text{(IX.74)}$$

where β is the vectorial angle (Fig. 85).

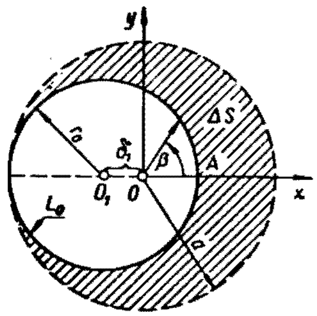

Fig. 85.

Then according to equality (IX.74) we have

$$\begin{gathered} \varepsilon_1(\beta) = a - R(\beta) = a - \\ - \sqrt{r_0^2 - \delta_1^2 \sin^2 \beta} + \delta_1 \cos \beta; \\ \varepsilon_1'(\beta) = \frac{\delta_1^2 \sin \beta \cos \beta}{\sqrt{r_0^2 - \delta_1^2 \sin^2 \beta}} - \delta_1 \sin \beta \\ (a = r_0 + \delta_1). \end{gathered} \qquad \text{(IX.75)}$$

Functions $\psi(a, \beta)$ and $\psi_r'(a, \beta)$ for the examined problem are represented by formulas (IX.70).

Putting expressions (IX.70) and (IX.75) in equation (IX.60), we obtain the formula

$$P_* = P_*(\beta) = \frac{K\sqrt{2R(\beta)}}{\pi \left\{ \psi_0 - \frac{1}{2} \varepsilon_1(\beta)\, \psi_1 + \psi_0 J_1(\beta) \right\}}, \qquad \text{(IX.76)}$$

where

$$\psi_0 = \frac{1}{\pi^2(1-\nu)a}\left\{\frac{1-\nu}{1+n^2} + \frac{n^2}{(1+n^2)^2}\right\}, \tag{IX.77}$$

$$\psi_1 = \frac{2(2-\nu)n^4 - 6n^2 - 2(1-\nu)}{\pi^2 a^2(1-\nu)(1+n^2)^3}$$

$$\left(n = \frac{h}{a}\right);$$

$$I_1(\beta) = \frac{\delta_1^2}{4\pi a r_0}\int_0^{2\pi}\frac{\sin\alpha\cos\alpha}{\sqrt{1-\eta_2^2\sin^2\alpha}}\,\mathrm{ctg}\,\frac{\beta-\alpha}{2}\,d\alpha -$$

$$-\frac{\delta_1}{4\pi a}\int_0^{2\pi}\sin\alpha\,\mathrm{ctg}\,\frac{\beta-\alpha}{2}\,d\alpha \quad \left(\eta_2^2 = \frac{\delta_1^2}{r_0^2}\right). \tag{IX.78}$$

The first integral in equality (IX.78) is expressed through the combination of full elliptic integrals of the first, second and third kind, and the second integral is calculated elementarily. In particular, when $\beta = 0$ we find

$$I_1(0) = \frac{r_0}{\pi a}\left[(1-\eta_2^2)K(\eta_2) - E(\eta_2)\right] + \frac{\delta_1}{2a}. \tag{IX.79}$$

Using this equality, formula (IX.76) for point A of contour L_0 (Fig. 85) can be transformed to the form

$$P_*(0) = \frac{K\sqrt{2(r_0-\delta_1)}}{\pi\{\psi_0 - \delta_1\psi_1 + \psi_0 I_1(0)\}}, \tag{IX.80}$$

where values of ψ_0, ψ_1 and $I_1(0)$ are determined by the formulas (IX.77) and (IX.79).

7. Pure Bend of a Beam with Internal Plane Elliptic Crack

Let us assume that an infinitely long beam with cross section $2h_0 \times 2d_0$ is bent by constant bending moments m (Figs. 86 and 87). Let us refer the beam to rectangular system of cartesian coordinates Oxyz and assume that the z-axis coincides with the axis of the beam, coordinate axes x and y are parallel to its lateral faces, and bending

moments M act in planes parallel to the plane $x = 0$. Let us assume that in the beam in the zone of tensile stresses in plane $z = 0$ is an elliptic crack. For this problem it is necessary to determine the limit value of bending moments $M = M_*$.

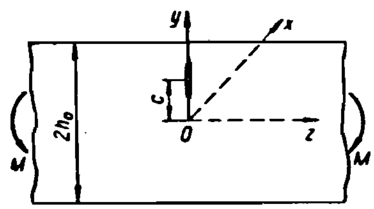

Fig. 86.

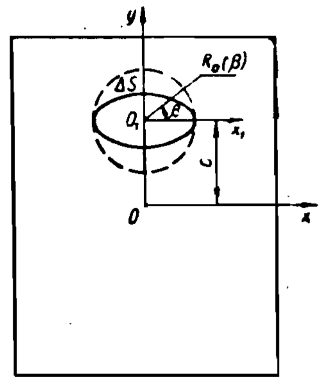

Fig. 87.

To simplify calculations we assume that the elliptic crack is oriented as shown on Fig. 87. In such a case the equation of the contour of the crack has the form

$$R_э(\beta) = \frac{b}{\sqrt{1 - k^2\cos^2\beta}} \quad \left(k^2 = \frac{a^2 - b^2}{a^2}\right), \tag{IX.81}$$

where a and b are the major and minor semiaxes of the crack which are small as compared to dimensions h_0, d_0 of the cross section of the beam.

Furthermore, we designate by c the distance of the center of the crack from plane $y = 0$, assuming that $c \geq a$.

To solve the formulated problem analogously to preceding examples we will inscribe around the crack in plane $z = 0$ a circle with radius a and calculate functions $\varepsilon_э(\beta)$, $\psi_*(a, \beta)$, $\psi_*'(a, \beta)$. In this case function $\varepsilon_э(\beta)$ is calculated by formula (IX.62), and in order to determine functions $\psi_*(a, \beta)$ and $\psi_*'(a, \beta)$ let us note that for the examined problem elastic stresses $\sigma_z(r, \beta, 0) = q(r, \beta)$ in a beam without a crack are expressed by the formula

$$q(r, \beta) = \frac{M}{J}(r \sin\beta + c), \tag{IX.82}$$

where $J = \frac{4}{3}h_0^3 d_0$ is the moment of inertia of the cross section of the beam with respect to the x-axis.

Using expression (IX.82) and formula (IX.6), we obtain

$$\psi(r, \beta) = \frac{M}{\pi^2 J}\int_0^a \int_0^{2\pi} \frac{\sqrt{a^2-\rho^2}(\rho \sin\alpha + c)\rho d\rho d\alpha}{r^2+\rho^2-2r\rho\cos(\alpha-\beta)} + \sqrt{r^2-a^2}\,q(r, \beta). \tag{IX.83}$$

To calculate the integrals in this formula we note (see formula (VI.29)) that

$$\int_0^{2\pi} \frac{\sin n\alpha d\alpha}{r^2+\rho^2-2r\rho\cos(\alpha-\beta)} =$$
$$= \frac{2\pi}{r^2+\rho^2}\cdot\frac{(1-\sqrt{1-\mu^2})^n}{\mu^n\sqrt{1-\mu^2}}\sin n\beta \; (\rho < r);$$

$$\int_0^{2\pi} \frac{d\alpha}{r^2+\rho^2-2r\rho\cos(\alpha-\beta)} = \frac{2\pi}{r^2+\rho^2}\cdot\frac{1}{\sqrt{1-\mu^2}} = \frac{2\pi}{r^2-\rho^2} \quad \left(\mu = \frac{2r\rho}{r^2+\rho^2}\right).$$

Using these formulas and equality (IX.83), we obtain

$$\psi(r, \beta) = \frac{2M}{\pi J}\left\{\frac{\sin\beta}{r}\int_0^a \frac{\sqrt{a^2-\rho^2}\rho^3 d\rho}{r^2-\rho^2} + c\int_0^a \frac{\sqrt{a^2-\rho^2}\rho d\rho}{r^2-\rho^2}\right\} +$$
$$+ \sqrt{r^2-a^2}\,q(r, \beta) \quad (r \geqslant a). \tag{IX.84}$$

Calculating integrals in equality (IX.84), and carrying out necessary transformations as $r \to a$, we obtain

$$\psi_*(a, \beta) = \frac{2M_*}{\pi J}\left(\frac{2}{3}a^2 \sin\beta + ac\right), \; \psi'_{*r}(a, \beta) = \frac{2M_*}{\pi J}(4a \sin\beta + 2c). \tag{IX.85}$$

After substituting in equation (IX.60) the values of $\varepsilon_э(\beta)$, $\psi_*(a, \beta)$, $\psi'_{*r}(a, \beta)$, represented by formulas (IX.62) and (IX.85), we obtain the following expression for calculation of the limit values of bending moment $M = M_*$:

$$M_*(\beta) = \frac{K\sqrt{2R_3(\beta)}}{\Lambda(\beta)}. \qquad (IX.86)$$

Here

$$\Lambda(\beta) = \frac{2}{J}\left\{\frac{2}{3}a^2\sin\beta + ac - (2a\sin\beta + c)\left(a - \frac{b}{\sqrt{1-k^2\cos^2\beta}}\right) + \right.$$
$$\left. + cI(\beta) + \frac{2}{3}a\left[aI_2(\beta) - b(1-k^2)I_3(\beta)\right]\right\}, \qquad (IX.87)$$

where

$$I_2(\beta) = \frac{1}{4\pi}\int_0^{2\pi}\cos\alpha\,\mathrm{ctg}\frac{\beta-\alpha}{2}\,d\alpha = \frac{1}{2}\sin\beta;$$
$$I_3(\beta) = \frac{1}{4\pi}\int_0^{2\pi}\frac{\cos\alpha}{(1-k^2\cos^2\alpha)^{3/2}}\,\mathrm{ctg}\frac{\beta-\alpha}{2}\,d\alpha.$$

In formula (IX.87) the integral $I_3(\beta)$ is expressed through the combination of full elliptic integrals according to the formula

$$I_3(\beta) = \frac{\sin\beta}{\pi(1-k^2\cos^2\beta)}\left\{\frac{E(k)}{1-k^2} - \Pi(n,\ k)\right\};\quad I_3\left(\frac{\pi}{2}\right) = \frac{E(k)}{\pi(1-k^2)}$$

(for designations see pp. 252-254).

We calculate values of M_* for point $A\left(\frac{1}{2}\pi\right)$ of the contour of the examined crack (see Fig. 87). Considering in equality (IX.86) that $\beta = \frac{1}{2}\pi$, we find

$$M_*\left(\frac{\pi}{2}\right) = \frac{K\sqrt{2b}}{\Lambda\left(\frac{1}{2}\pi\right)}, \qquad (IX.88)$$

where

$$\Lambda\left(\frac{\pi}{2}\right) = \frac{2}{J}\left\{2ab - a^2 + bc + \frac{b}{\pi}\left[cK(k) - \left(c + \frac{2}{3}a\right)E(k)\right]\right\}.$$

In the special case when the contour of the examined crack (Fig. 87) is a circle with radius a, i.e., when a = b, from formula (IX.88) we obtain

$$M_*\left(\frac{\pi}{2}\right) = \frac{3JK}{(2a+3c)\sqrt{2a}}. \qquad (IX.89)$$

Using formulas (IX.88) and (IX.89) in each concrete case one can determine the limit value of the pure bend of a beam with internal crack, which in its plane is in the shape of an ellipse or circle.

8. The Breaking Load During the Extension of a Brittle Body with a Plane Crack Nearly Circular in Shape

During the investigation of maximum equilibrium of brittle bodies with cracks, determination of the limit load has an important value, i.e., the load after which spontaneous propagation of the crack sets in, leading to full rupture of the body. As already was noted, such a load does not always coincide with the maximum.

Below according to the results of this chapter a problem is formulated about approximation calculation of the breaking load for a brittle body with a crack nearly circular in shape when the body is subjected to uniaxial extension perpendicular to the plane of the crack [121]. As example the problem is examined for the case of an elliptic crack.

Formulation of the problem. Let us assume that a three-dimensional brittle body weakened by crack S_0 (Fig. 88) nearly circular in shape is subjected to extension by external forces Q, applied to the body symmetrically with respect to the plane of location of the crack and increasing proportionally to certain parameter λ'. For such a problem it is required to determine the dimensions (form) of the crack for every value of $\lambda' > 0$, i.e., to determine the kinetics of propagation of crack S_0 during the growth of this parameter, and also to find the value of parameter $\lambda = \lambda_*'$ after which propagation of the moving-equilibrium crack S_0 becomes spontaneous and the body ruptures.

In the process of monotonic increase of load $Q = \lambda' Q_0$, applied to the examined brittle body with plane crack S_0 (see Fig. 88), this load at a certain value of parameter λ' reaches the limit value $\lambda' Q_0 = Q_*$. Let us assume that for the limit load parameter $\lambda' = 1$.

With such a load ($Q = Q_*$, $\lambda' = 1$) at least in one point of contour L_0 (these points can be only a few) the maximum-equilibrium state of the crack sets in, and consequently, in these points equality (IX.60) holds.

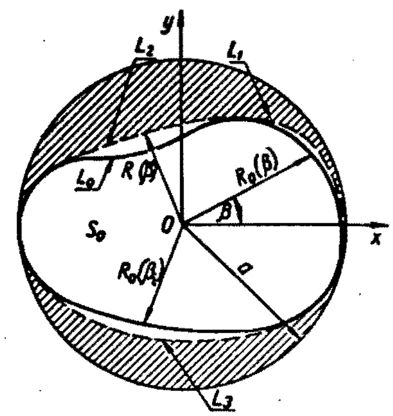

Fig. 88.

On the basis of formula (IX.60), obviously one can determine the value of limit load $Q = Q_*$ for a brittle body weakened by crack S_0 (see Fig. 88). However this is only the first stage in solving the formulated problem. Full resolution of the problem consists in determination of breaking load $Q = Q_{**}$, i.e., load after which crack propagation leads to complete rupture of the body. For determination of load Q_{**} and form of the developed crack S_0 when $Q_* < Q < Q_{**}$ we will note the following.

After external load Q reaches Q_* contour L_0 of the initial crack S_0 in the neighborhood of points $R_0(\beta_i)$ will go into the moving-equilibrium state and with a subsequent (even small) increase of load $Q > Q_*$ in the neighborhood of points $R_0(\beta_i)$ will start to shift into the plane of location of the crack.

We pose the problem: to determine contour L (dotted line on Fig. 88) of moving-equilibrium crack S, formed from crack S_0 as a result of monotonic increase of load Q to the level

$$Q = \lambda' Q_* \text{ when } \lambda' > 1. \tag{IX.90}$$

The radius vector of contour L can be represented in the form

$$R(\beta) = R_0(\beta) + \varepsilon_1(\beta), \tag{IX.91}$$

where $\varepsilon_1(\beta)$ – unknown continuous function of argument β and parameter λ'.

If function $\varepsilon_1(\beta)$ is determined for a preassigned configuration of contour L_0 of initial crack S_0 and a given value of parameter λ', the problem about contour L of moving-equilibrium crack S will be solved.

Arcs L_j ($j = 1, 2, 3...$) of contour L, not coinciding with initial contour L_0, are in state of dynamic equilibrium when $Q = \lambda Q_*$, where $\lambda \geq 1$, therefore for these arcs condition (IX.60) should be fulfilled, where function $\varepsilon(\beta)$ for arc L_j according to (IX.91) is expressed so:

$$\varepsilon(\beta) = a - R(\beta) = \varepsilon_0(\beta) - \varepsilon_1(\beta), \tag{IX.92}$$

where $\varepsilon_0(\beta)$ is a known function $\varepsilon_0(\beta) = a - R_0(\beta)$.

Putting expressions (IX.91) and (IX.92) into (IX.60), we find the equation for determination of function $\varepsilon_1(\beta)$

$$\begin{gathered}
\frac{\pi}{\sqrt{2[R_0(\beta)+\varepsilon_1(\beta)]}}\Big\{\psi_*(a,\beta) - \frac{1}{2}\varepsilon_0(\beta)\psi'_{*r}(a,\beta) + \\
+\frac{1}{2}\varepsilon_1(\beta)\psi'_{*r}(a,\beta) + \frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{d\alpha}[\varepsilon_0(\alpha)\psi_*(a,\alpha)]\operatorname{ctg}\frac{\beta-\alpha}{2}d\alpha - \\
-\frac{1}{4\pi a}\sum_{j=1}^{m}\int_{\beta_{1j}}^{\beta_{2j}}\frac{d}{d\alpha}[\varepsilon_1(\alpha)\psi_*(a,\alpha)]\operatorname{ctg}\frac{\beta-\alpha}{2}d\alpha\Big\} = K,
\end{gathered} \tag{IX.93}$$

where β_{1j} and β_{2j} are vectorial angles corresponding to initial and end points of arc L_j, $\beta_{1j} \leq \beta \leq \beta_{2}j$.

On the ends of moving-equilibrium sections L_j of contour L function $\varepsilon_1(\beta)$ should satisfy conditions reflecting the fact that for points of arcs L_j ($j = 1, 2, 3...$) function $\varepsilon_1(\beta)$ is continuous and positive, and on arcs $L - L_j$, i.e., on sections of L coinciding with initial contour L_0, $\varepsilon_1(\beta) \equiv 0$. Furthermore, arcs L_j smoothly pass into the contour of the initial crack.[1] Because of this, function $\varepsilon_1(\beta)$ should satisfy the conditions

$$\begin{gathered}\varepsilon_1(\beta_{1j}) = \varepsilon_1(\beta_{2j}) = 0 \quad (j = 1,\ 2,\ 3\ldots); \\ \left[\frac{d\varepsilon_1(\beta)}{d\beta}\right]_{\beta=\beta_{1j}} = \left[\frac{d\varepsilon_1(\beta)}{d\beta}\right]_{\beta=\beta_{2j}} = 0.\end{gathered} \tag{IX.94}$$

Equation (IX.60) and, consequently, equation (IX.93) are composed accurate to quantities whose order of smallness is $\frac{\varepsilon(\beta)}{a}$, where a is the radius of the circle inscribed around the contour of crack S or S_0.

Function $\varepsilon_1(\beta)$, which can be determined from equation (IX.93), satisfies inequality $0 \leq \varepsilon_1(\beta) \leq \varepsilon_0(\beta)$ (see Fig. 88), i.e., it is a function of the same order of smallness compared with a as function $\varepsilon_0(\beta)$. Therefore, without disturbing the accuracy of equation (IX.93), it can be simplified by keeping only terms linear relatively to $\varepsilon_1(\beta)$ and $\varepsilon_0(\beta)$. After simplifications this equation accurate to small $\frac{\varepsilon_1(\beta)}{a}$ can be presented in the following form:

$$\begin{gathered}f_0(a,\ \beta)\,\varepsilon_1(\beta) - \frac{1}{4\pi a}\sum_{j=1}^{m}\int_{\beta_{1j}}^{\beta_{2j}}\frac{d}{d\alpha}\left[\varepsilon_1(\alpha)\,\psi_*(a,\ \alpha)\right] \times \\ \times \operatorname{ctg}\frac{\beta-\alpha}{2}\,d\alpha = f_1(a,\ \lambda',\ \beta),\end{gathered} \tag{IX.95}$$

where

$$\begin{gathered}f_0(a,\ \beta) = \frac{1}{2}\left\{\psi'_{*,r}(a,\ \beta) - \frac{2K}{\pi\sqrt{2R_0(\beta)}}\right\} = \\ = \frac{1}{2}\left\{\psi'_{*,r}(a,\ \beta) - \frac{\psi_*(a,\ \beta)}{a} + 0\,(\varepsilon_1\varepsilon_0)\right\};\end{gathered} \tag{IX.96}$$

---

[1]Evenness of transition of arcs L_j into contour L_0 is a requirement for the absence of angular points on contour L.

$$f_1(a, \lambda', \beta) = \frac{K}{\pi}\sqrt{2R_0(\beta)} - \psi_*(a, \beta) + \frac{1}{2}\varepsilon_0(\beta)\psi'_{*r}(a, \beta) -$$

$$-\frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{d\alpha}[\varepsilon_0(\alpha)\psi_*(a, \alpha)]\,\mathrm{ctg}\,\frac{\beta-\alpha}{2}\,d\alpha; \qquad \text{(IX.96 Cont'd)}$$

$\psi_*(a, \beta)$, $\psi'_{*r}(a, \beta)$ – values of the functions $\psi(a, \beta)$, $\psi'_r(a, \beta)$ respectively when $Q = \lambda' Q_*$.

Equation (IX.95) is initial for determination of moving-equilibrium crack S (see Fig. 88) during monotonic growth of external load $\lambda' Q_*$, when $\lambda' > 1$. Solving this equation with respect to function $\varepsilon_1(\beta)$ for a preassigned configuration of crack $R_0(\beta)$ and $\lambda' \geq 1$, and then, satisfying conditions (IX.94), we find finally function $\varepsilon_1(\beta)$ and the dependence of angles $\beta_{1.j}$ and $\beta_{2.j}$ on parameter λ'. Using further the expression $\varepsilon_1(\beta)$ and equality (IX.91), it is possible to trace the kinetics of propagation of a nearly circular crack while parameter λ' increases from $\lambda' = 1$ to $\lambda' = \lambda'_*$, where λ'_* is the limit (maximum) value of parameter $\lambda' \geq 1$ at which a solution exists for equation (IX.95) and the following inequality holds:

$$\max\{R_0(\beta) + \varepsilon_1(\beta)\} \leqslant a. \qquad \text{(IX.97)}$$

If external load $\lambda' Q_*$ constitutes uniform extension to infinity and reaches $\lambda'_* Q_*$, then further propagation of the examined crack due to what was said above becomes spontaneous, the whole contour of the crack becomes moving-equilibrium and the body ruptures.

Thus, in this case of breaking load $Q = Q_{**}$ for a body weakened by a plane isolated crack, nearly circular in shape, is determined by the equality

$$Q_{**} = \lambda'_* Q_*. \qquad \text{(IX.98)}$$

As an example we will determine the breaking load for a brittle body with a plane elliptical crack (Fig. 89). Let us assume that such a body at infinity is subjected to extension by monotonically increasing stresses $\sigma_z(x, y, \infty) = p$.

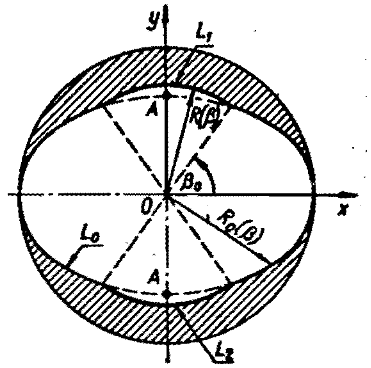

Fig. 89.

In this case (see Chap. VIII) the limit value of external stress is $p = p_*$ is determined by the formula

$$p_* = p_*^{(b)} = E(k)\frac{\sqrt{2}K}{\pi\sqrt{b}}, \tag{IX.99}$$

where $k^2 = 1 - k_1^2$; $k_1 = \frac{b}{a}$: a and b – major and minor semiaxis of elliptic crack respectively; E(k) – full elliptic integral of the second kind.

If external load p reaches $P_*^{(b)}$, in points located on the minor axis of the ellipse (points A on Fig. 89), the contour of the crack goes into a state of dynamic equilibrium. Stress p after reaching $p_*^{(b)}$ increases proportionally to parameter λ', i.e.,

$$p = \lambda' p_*^{(b)} = \lambda' p_* \quad (\lambda' \geqslant 1). \tag{IX.100}$$

We will determine for a given case the contour of a crack propagating across the cross section of a body, when parameter λ' monotonically increases, and also we will find the limit value of this parameter $\lambda' = \lambda_*'$, after which a body with an elliptic crack ruptures.

At a certain value of parameter $\lambda' = \lambda_i' < \lambda_*'$ the contour of a moving crack is characterized by arcs L_1 and L_2 (see Fig. 89). The formula for determination of radius-vector $R(\beta)$ of contour L can be presented in the form

$$R(\beta)=\begin{cases} R_0(\beta)+\varepsilon_1(\beta) & \text{on } L_1 \text{ and } L_2; \\ R_0(\beta) & \text{outside } L_1 \text{ and } L_2, \end{cases} \tag{IX.101}$$

where

$$R_0(\beta)=R_s(\beta)=\frac{b}{\sqrt{1-k^2\cos^2\beta}}. \tag{IX.102}$$

Using formulas (IX.92) and (IX.102), we find

$$\varepsilon_0(\beta)=a-\frac{b}{\sqrt{1-k^2\cos^2\beta}};\quad \frac{d\varepsilon_0(\beta)}{d\beta}=\frac{bk^2\sin^2\beta\cos\beta}{\sqrt{(1-k^2\cos^2\beta)^3}}. \tag{IX.103}$$

Functions $\psi_*(a, \beta)$ and $\psi'_{*_r}(a, \beta)$ for the examined form of external load $[\sigma_z(x, y, \infty) = \lambda'_i p_*]$ is determined according to the formula (IX.63a):

$$\psi_*(a, \beta)=\frac{2a\lambda_i' p_*}{\pi};\ \psi'_{*_r}(a, \beta)=\frac{4\lambda_i' p_*}{\pi}. \tag{IX.104}$$

Putting expressions (IX.102)-(IX.104) in equation (IX.95), we find

$$\varepsilon_1(\beta)-\frac{1}{2\pi}\sum_{j=1}^{2}\int_{L_j}\varepsilon_1'(\alpha)\operatorname{ctg}\frac{\beta-\alpha}{2}d\alpha=\bar f_1(a, \lambda_i', \beta). \tag{IX.105}$$

Here β is the vectorial angle of points of arcs L_j;

$$\varepsilon_1'(\alpha)=\frac{d\varepsilon_1(\alpha)}{d\alpha};$$

$$\bar f_1(a, \lambda_i', \beta)=\frac{\pi}{\lambda_i' p_*}f_1(a, \lambda_i, \beta)\cong b\left\{\frac{\pi}{\lambda_i' E(k)\sqrt[4]{1-k^2\cos^2\beta}}-\right.$$
$$\left.-\frac{2}{\sqrt{1-k^2\cos^2\beta}}-\frac{1}{2\pi}J(\beta)\right\}, \tag{IX.106}$$

where

$$J(\beta)=k^2\int_0^{2\pi}\frac{\sin\alpha\cos\alpha}{\sqrt{(1-k^2\cos^2\beta)^3}}\operatorname{ctg}\frac{\beta-\alpha}{2}d\alpha=$$
$$=4\left\{K(k)-\frac{E(k)}{1-k^2\cos^2\beta}-\frac{k^2\sin^2\beta}{1-k^2\cos^2\beta}\Pi(n, k)\right\}\left(n=-\frac{1}{\cos^2\beta}\right).$$

For the examined problem arcs L_1 and L_2 are symmetric relative to polar axis Ox, therefore equation (IX.105) can be written so:

$$\varepsilon_1(\beta) - \frac{1}{2\pi}\left\{\int_{\beta_0}^{\pi-\beta_0} \varepsilon_1'(\alpha)\operatorname{ctg}\frac{\beta-\alpha}{2}\,d\alpha + \int_{\pi+\beta_0}^{2\pi-\beta_0} \varepsilon_1'(\alpha) \times \right.$$
$$\left. \times \operatorname{ctg}\frac{\beta-\alpha}{2}\,d\alpha = \bar{f}_1(a, \lambda_i', \beta), \right. \qquad \text{(IX.107)}$$

where β_0 is the vectorial angle (see Fig. 89) corresponding to the beginning of arc L_1, $\beta_0 \leq \beta \leq \pi - \beta_0$, $\pi + \beta_0 \leq \beta \leq 2\pi - \beta_0$.

We assume that eccentricity k of the examined crack is small, so that quantities containing factor k^{2n} when $n \geq 2$ are disregarded, since they are considerably less than one. In such a case equation (IX.107) takes the form

$$\varepsilon_1(\beta) - \frac{1}{2\pi}\left\{\int_{\beta_0}^{\pi-\beta_0} \varepsilon_1'(\alpha)\operatorname{ctg}\frac{\beta-\alpha}{2}\,d\alpha + \right.$$
$$\left. + \int_{\pi+\beta_0}^{2\pi-\beta_0} \varepsilon_1'(\alpha)\operatorname{ctg}\frac{\beta-\alpha}{2}\,d\alpha\right\} = B(\beta). \qquad \text{(IX.108)}$$

Here

$$B(\beta) = B_0 + B_1\cos^2\beta,$$

where

$$B_0 = b\left(\frac{1}{\lambda_i'} - 1\right)\frac{4+k^2}{2}; \quad B_1 = \frac{bk^2}{2\lambda_i'}. \qquad \text{(IX.109)}$$

Considering in equation (IX.108) that $\beta = \frac{\pi}{2} + \vartheta$ and $\alpha = \frac{\pi}{2} + \vartheta_1$, and also taking into account that for the examined example function $\varepsilon_1(\beta)$ has period π, we find

$$\varepsilon_1[\vartheta] - \frac{1}{\pi}\int_{-\vartheta_0}^{\vartheta_0} \varepsilon_1'[\vartheta_1]\operatorname{ctg}(\vartheta - \vartheta_1)\,d\vartheta_1 = B_0 + B_1\sin^2\vartheta, \qquad \text{(IX.110)}$$

where

$$\varepsilon_1[\vartheta] = \varepsilon_1\left(\frac{\pi}{2} + \beta\right); \quad \vartheta_0 = \frac{\pi}{2} - \beta_0, \quad -\vartheta_0 \leqslant \vartheta \leqslant \vartheta_0.$$

Considering the construction of an approximate solution of equation (IX.110), we will replace in this equation variables ϑ and ϑ_1, considering

$$\operatorname{tg} \vartheta_1 = t, \quad \operatorname{tg} \vartheta = x, \quad \operatorname{tg} \vartheta_0 = x_0. \tag{IX.111}$$

As a result of such transformations we obtain

$$g(x) - \frac{1+x^2}{\pi} \int_{-x_0}^{x_0} \frac{g'(t)}{x-t}\, dt = B_0 + \frac{B_1 x^2}{1+x^2}, \tag{IX.112}$$

where

$$-x_0 \leqslant x \leqslant x_0; \quad g(x) = \varepsilon_1 [\operatorname{arctg} x].$$

For function g(x) conditions (IX.94) have the form

$$g(\pm x_0) = 0; \quad g'_x(\pm x_0) = 0. \tag{IX.113}$$

We seek an approximate solution of equation (IX.112) in the following form [37, 109, 171]:

$$g(x) \approx \sqrt{1 - \frac{x^2}{x_0^2}} \left(c_0 + c_2 \frac{x^2}{x_0^2}\right). \tag{IX.114}$$

For determination of coefficients c_0 and c_2 we place expression (IX.114) in the left side of equation (IX.112) and assume [109] that obtained the expression equals the left side of equation (IX.114) in points $x = 0$ and $x = \frac{1}{2}x_0$. As a result we obtained the following system of equations:

$$\begin{gathered} c_0 m_0(0) + c_2 m_2(0) = B_0; \\ c_0 m_0\left(\frac{1}{2} x_0\right) + c_2 m_2\left(\frac{1}{2} x_0\right) = B_2, \end{gathered} \tag{IX.115}$$

where

$$B_2 = \frac{4}{4+x_0^2}\left(B_0 + \frac{B_1 x_0^2}{4+x_0^2}\right); \quad m_0(x) = \frac{\sqrt{1-\left(\frac{x}{x_0}\right)^2}}{1+x^2} - \frac{1}{x_0};$$

$$m_2(x) = \frac{x^2}{x_0^2(1+x^2)} \sqrt{1-\left(\frac{x}{x_0}\right)^2} - \frac{1}{x_0}\left(3\frac{x^2}{x_0^2} - \frac{1}{2}\right). \tag{IX.115a}$$

Solving this system, we find

$$c_0 = \frac{D_0}{D}; \quad c_2 = \frac{D_2}{D}, \tag{IX.116}$$

where

$$D_0 = \frac{\sqrt{3}\,B_0}{2(4+x_0^2)} - \frac{B_0 + 2B_1}{4x_0};$$

$$D = \frac{\sqrt{3}}{2(4+x_0^2)} - \frac{6\sqrt{3} + 4 + x_0^2}{4x_0(4+x_0^2)} + \frac{3}{4x_0^2};$$

$$D_2 = B_1\left(1 - \frac{1}{x_0}\right) - B_0\left(\frac{2\sqrt{3}}{4+x_0^2} - \frac{1}{x_0}\right).$$

So that function g(x), represented by formula (IX.114), satisfied boundary conditions (IX.113), the equation $c_0 + c_2 = 0$ must hold. This reflects the connection between parameters λ_i' and x_0. According to expression (IX.116) it has the form

$$B_0\left[\frac{4}{4+x_0^2}\left(1 - \frac{3}{2x_0}\right) + \right.$$
$$\left. + \frac{3}{4x_0} - \frac{3\sqrt{3}}{2(4+x_0^2)}\right] +$$
$$+ B_1 \frac{4x_0^2}{(4+x_0^2)^2}\left(1 - \frac{3}{2x_0}\right) = 0, \tag{IX.117}$$

where parameters B_0 and B_1 are represented by formulas (IX.109).

On the basis of relationships (IX.109) and (IX.117) accurate to quantities of the order $0(k^4)$ we find

$$\lambda_i' = 1 + 0{,}25k^2\lambda_0'(x_0) + 0(k^4), \tag{IX.118}$$

where

$$\lambda_0'(x_0) = \frac{8x_0^2(2x_0 - 3)}{(4+x_0^2)\,[3x_0^2 + 2(8 - 3\sqrt{3})\,x_0 - 12]};$$

$$x_0 = \operatorname{tg}\vartheta_0 = \operatorname{tg}\left(\frac{\pi}{2} - \beta_0\right), \; x_0 \geqslant 0. \tag{IX.119}$$

The graph of the change of function $\lambda_0'(x_0)$ as parameter x_0 grows is shown on Fig. 90. According to this graph, the propagation of an elliptic crack after external stresses reach $\lambda_i' p_*^{(b)}$, where $\lambda_i > 1$ at the beginning is unstable, and then (when $x_0 > \sim 1.5$) is stable. For this problem the limit value of parameter $\lambda_i' = \lambda_*'$ is determined approximately on the basis of expression (IX.118) and the graph on Fig. 90 by the formula

$$\lambda_*' \approx 1 + 0{,}14k^2. \tag{IX.120}$$

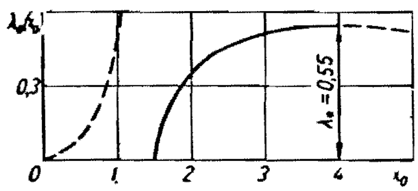

Fig. 90.

Thus, the breaking load for a brittle body weakened by an internal plane elliptic crack, when such a body is subjected at infinity to extension by monotonically increasing stresses p, directed perpendicularly to the plane of the crack, is determined by the formula

$$p_{**} \approx (1 + 0{,}14k^2)\, p_*^{(b)}.$$

Hence, in particular, it follows that for the examined body the lebel of rupture stresses p_{**} exceeds the level of limit stresses $p_*^{(b)}$ by an insignificant amount if eccentricity k of the elliptic crack is smaller than one unit. In such a case we can practically consider that p_{**} and $p_*^{(b)}$ coincide.

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ABSTRACT

(U) This book is intended for scientists, engineers, and technicians dealing with strength problems of solids. The monograph analyzes the theoretical principles of a quasistationary equilibrium of brittle solids with fractures. The results of investigations which were carried out by the author on the theory of propagation of fractures in a deformable solid are generalized and calculated model diagrams for such problems are given. Solutions of new plane and three-dimensional problems on a limiting equilibrium of brittle solids with fracture are shown. An attempt is made to formulate the elements of the theory of brittle destruction of deformable solids with fracture defects. References - 238

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